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## Stabilization and controllability of first-order integro-differential hyperbolic equations



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### ABSTRACT

In the present article we study the stabilization of first-order linear integro-differential hyperbolic equations. For such equations we prove that the stabilization in finite time is equivalent to the exact controllability property. The proof relies on a Fredholm transformation that maps the original system into a finite-time stable target system. The controllability assumption is used to prove the invertibility of such a transformation. Finally, using the method of moments, we show in a particular case that the controllability is reduced to the criterion of Fattorini.

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### 1. Introduction and main results

The purpose of this article is the study of the stabilization and controllability properties of the equation

$$\left\{ \begin{array}{l} u_t(t, x) - u_x(t, x) = \int_0^L g(x, y)u(t, y) dy, \quad t \in (0, T), x \in (0, L), \\ u(t, L) = U(t), \quad t \in (0, T), \\ u(0, x) = u^0(x), \quad x \in (0, L). \end{array} \right. \tag{1.1}$$

In (1.1),  $T > 0$  is the time of control,  $L > 0$  the length of the domain.  $u^0$  is the initial data and  $u(t, \cdot) : [0, L] \rightarrow \mathbb{C}$  is the state at time  $t \in [0, T]$ ,  $g : (0, L) \times (0, L) \rightarrow \mathbb{C}$  is a given function in  $L^2((0, L) \times (0, L))$  and, finally,  $U(t) \in \mathbb{C}$  is the boundary control at time  $t \in (0, T)$ .

The stabilization and controllability of (1.1) started in [13]. The authors proved that the equation

$$\left\{ \begin{array}{l} u_t(t, x) - u_x(t, x) = \int_0^x g(x, y)u(t, y) dy + f(x)u(t, 0), \quad t \in (0, T), x \in (0, L), \\ u(t, L) = U(t), \quad t \in (0, T), \\ u(0, x) = u^0(x), \quad x \in (0, L), \end{array} \right.$$

with  $g$  and  $f$  continuous, is always stabilizable in finite time. The proof uses the backstepping approach introduced and developed by M. Krstic and his co-workers (see, in particular, the pioneer articles [2,16,19] and the reference book [14]). This approach consists in mapping (1.1) into the following finite-time stable target system

$$\left\{ \begin{array}{l} w_t(t, x) - w_x(t, x) = 0, \quad t \in (0, T), x \in (0, L), \\ w(t, L) = 0, \quad t \in (0, T), \\ w(0, x) = w^0(x), \quad x \in (0, L), \end{array} \right.$$

by means of the Volterra transformation of the second kind

$$u(t, x) = w(t, x) - \int_0^x k(x, y)w(t, y)dy, \tag{1.2}$$

where the kernel  $k$  has to satisfy some PDE in the triangle  $0 \leq y \leq x \leq L$  with appropriate boundary conditions, the so-called kernel equation. Let us emphasize that the strength of this method is that the Volterra transformation (1.2) is always invertible

(see e.g. [10, Chapter 2, THEOREM 6]). Now, if the integral term is not anymore of Volterra type, that is if  $g$  in (1.1) does not satisfy

$$g(x, y) = 0, \quad x \leq y, \tag{1.3}$$

then, the Volterra transformation (1.2) can no longer be used (there is no solution to the kernel equation which is supported in the triangle  $0 \leq y \leq x \leq L$  in this case, see the equation (2.16) below). In [3], the authors suggested to replace the Volterra transformation (1.2) by the more general Fredholm transformation

$$u(t, x) = w(t, x) - \int_0^L k(x, y)w(t, y)dy, \tag{1.4}$$

where  $k \in L^2((0, L) \times (0, L))$  is a new kernel. However, the problem is now that, unlike the Volterra transformation (1.2), the Fredholm transformation (1.4) is not always invertible. In [3], the authors proved that, if  $g$  is small enough, then the transformation (1.4) is indeed invertible, see [3, Theorem 9]. They also gave some sufficient conditions in the case  $g(x, y) = g(y)$ , see [3, Theorem 1.11]. Our main result states that we can find a particular kernel  $k$  such that the corresponding Fredholm transformation (1.4) is invertible, if we assume that (1.1) is exactly controllable at time  $L$ . Finally, let us point out that Fredholm transformations have also been used to prove the exponential stabilization for a Korteweg–de Vries equation in [5] and for a Kuramoto–Sivashinsky equation in [6]. In these papers also, the existence of the kernel and the invertibility of the associated transformation were established under a controllability assumption. However, our proof is of a completely different spirit than the one given in these articles.

1.1. *Well-posedness*

Multiplying formally (1.1) by the complex conjugate of a smooth function  $\bar{\phi}$  and integrating by parts, we are lead to the following definition of solution:

**Definition 1.1.** Let  $u^0 \in L^2(0, L)$  and  $U \in L^2(0, T)$ . We say that a function  $u$  is a (weak) solution to (1.1) if  $u \in C^0([0, T]; L^2(0, L))$  and

$$\begin{aligned} & \int_0^\tau \int_0^L u(t, x) \overline{\left( -\phi_t(t, x) + \phi_x(t, x) - \int_0^L \overline{g(y, x)}\phi(t, y) dy \right)} dx dt \\ & + \int_0^L u(\tau, x)\overline{\phi(\tau, x)} dx - \int_0^L u^0(x)\overline{\phi(0, x)} dx - \int_0^\tau U(t)\overline{\phi(t, L)} dt = 0, \end{aligned} \tag{1.5}$$

for every  $\phi \in C^1([0, \tau] \times [0, L])$  such that  $\phi(\cdot, 0) = 0$ , and every  $\tau \in [0, T]$ .

Let us recall that (1.1) can equivalently be rewritten in the abstract form

$$\begin{cases} \frac{d}{dt}u = Au + BU, & t \in (0, T), \\ u(0) = u^0, \end{cases} \tag{1.6}$$

where we can identify the operators  $A$  and  $B$  through their adjoints by taking formally the scalar product of (1.6) with a smooth function  $\phi$  and then comparing with (1.5). The operator  $A : D(A) \subset L^2(0, L) \rightarrow L^2(0, L)$  is thus given by

$$Au = u_x + \int_0^L g(\cdot, y)u(y) dy, \tag{1.7}$$

with

$$D(A) = \{ u \in H^1(0, L) \mid u(L) = 0 \}.$$

Clearly,  $A$  is densely defined, and its adjoint  $A^* : D(A^*) \subset L^2(0, L) \rightarrow L^2(0, L)$  is

$$A^*z = -z_x + \int_0^L \overline{g(y, \cdot)}z(y) dy, \tag{1.8}$$

with

$$D(A^*) = \{ z \in H^1(0, L) \mid z(0) = 0 \}.$$

Using the Lumer–Philips’ theorem (see e.g. [18, Chapter 1, Corollary 4.4]), we can prove that  $A$  generates a  $C_0$ -semigroup  $(S(t))_{t \geq 0}$ .

Since  $A^*$  is closed, its domain  $D(A^*)$  is then a Hilbert space, equipped with the scalar product associated with the graph norm  $\|z\|_{D(A^*)} = (\|z\|_{L^2}^2 + \|A^*z\|_{L^2}^2)^{1/2}$ ,  $z \in D(A^*)$ . Observe that

$$\|\cdot\|_{D(A^*)} \text{ and } \|\cdot\|_{H^1(0,L)} \text{ are equivalent norms on } D(A^*). \tag{1.9}$$

On the other hand, the operator  $B \in \mathcal{L}(\mathbb{C}, D(A^*))'$  is

$$\langle BU, z \rangle_{D(A^*)', D(A^*)} = U\overline{z(L)}. \tag{1.10}$$

Note that  $B$  is well defined since  $BU$  is continuous on  $H^1(0, L)$  (by the trace theorem  $H^1(0, L) \hookrightarrow C^0([0, L])$ ) and since we have (1.9). Its adjoint  $B^* \in \mathcal{L}(D(A^*), \mathbb{C})$  is

$$B^*z = z(L). \tag{1.11}$$

One can prove that  $B$  satisfies the following so-called admissibility condition<sup>4</sup>:

$$\exists C > 0, \int_0^T |B^*S(T-t)^*z|^2 dt \leq C \|z\|_{L^2(0,L)}^2, \quad \forall z \in D(A^*). \tag{1.12}$$

Note that  $B^*S(T-\cdot)^*z$  makes sense in (1.12) since  $S(T-\cdot)^*z \in D(A^*)$  for  $z \in D(A^*)$ , while it does not in general if  $z$  is only in  $L^2(0, L)$ . Thus, (1.12) allows us to continuously extend in a unique way the map  $z \mapsto B^*S(T-\cdot)^*z$  to the whole space  $L^2(0, L)$  and give in particular a sense to  $B^*S(T-\cdot)^*z$  for  $z \in L^2(0, L)$ . We shall keep the same notation to denote this extension.

Finally, we recall that, since  $A$  generates a  $C_0$ -semigroup and  $B$  is admissible, for every  $u^0 \in L^2(0, L)$  and every  $U \in L^2(0, T)$ , there exists a unique solution  $u \in C^0([0, T]; L^2(0, L))$  to (1.1). Moreover, there exists  $C_T > 0$  (which does not depend on  $u^0$  nor  $U$ ) such that

$$\|u\|_{C^0([0,T];L^2(0,L))} \leq C_T \left( \|u^0\|_{L^2(0,L)} + \|U\|_{L^2(0,T)} \right).$$

See e.g. [4, Theorem 2.37] and [4, Section 2.3.3.1].

### 1.2. Controllability and stabilization

Let us now recall the definitions of the properties we are interested in.

**Definition 1.2.** We say that (1.1) is exactly controllable at time  $T$  if, for every  $u^0, u^1 \in L^2(0, L)$ , there exists  $U \in L^2(0, T)$  such that the corresponding solution  $u$  to (1.1) satisfies

$$u(T) = u^1.$$

If the above property holds for  $u^1 = 0$ , we say that (1.1) is null-controllable at time  $T$ .

**Remark 1.** For generators of  $C_0$ -group, exact and null controllability are equivalent properties. In our case  $A$  does not generate a group but these properties are still equivalent for (1.1). This follows from Proposition 3.1 below.

**Definition 1.3.** We say that (1.1) is stabilizable in finite time  $T$  if there exists a bounded linear map  $\Gamma : L^2(0, L) \rightarrow \mathbb{C}$  such that, for every  $u^0 \in L^2(0, L)$ , the solution  $u \in C^0([0, +\infty); L^2(0, L))$  to

---

<sup>4</sup> The proof is analogous to the one of Lemma C.2 in Appendix C.

$$\begin{cases} u_t(t, x) - u_x(t, x) = \int_0^L g(x, y)u(t, y) dy, & t \in (0, +\infty), x \in (0, L), \\ u(t, L) = \Gamma u(t), & t \in (0, +\infty), \\ u(0, x) = u^0(x), & x \in (0, L), \end{cases} \tag{1.13}$$

satisfies

$$u(t) = 0, \quad \forall t \geq T. \tag{1.14}$$

Note that (1.13) is well-posed. Indeed, by the Riesz representation theorem, there exists  $\gamma \in L^2(0, L)$  such that

$$\Gamma u = \int_0^L u(y)\overline{\gamma(y)} dy, \tag{1.15}$$

and (1.13) with (1.15) is well-posed.

**Remark 2.** Let us recall here some links between stabilization and controllability. Clearly, stabilization in finite time  $T$  implies null-controllability at time  $T$ . It is also well-known that in finite dimension (that is when  $A$  and  $B$  are matrices) controllability is equivalent to exponential stabilization at any decay rate, see e.g. [22, PART I, Theorem 2.9]. Finally, for bounded operators  $B$  (which is not the case here though), null-controllability at some time implies exponential stabilization, see e.g. [22, PART IV, Theorem 3.3]. We refer to [11] and the references therein for recent results on the exponential stabilization of one-dimensional systems generated by  $C_0$ -groups and to [1] for the exponential stabilization of systems generated by analytic  $C_0$ -semigroups.

### 1.3. Main results

Let us introduce the triangles

$$\begin{aligned} \mathcal{T}_- &= \{(x, y) \in (0, L) \times (0, L) \mid x > y\}, \\ \mathcal{T}_+ &= \{(x, y) \in (0, L) \times (0, L) \mid x < y\}. \end{aligned}$$

For the stabilization, we will always assume that

$$g \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+). \tag{1.16}$$

This means that we allow integral terms whose kernel has a discontinuity along the diagonal of the square  $(0, L) \times (0, L)$ :

$$\int_0^x g_1(x, y)u(t, y) dy + \int_x^L g_2(x, y)u(t, y) dy,$$

with  $g_1, g_2 \in H^1((0, L) \times (0, L))$ . We gathered in [Appendix A](#) some properties of the functions of  $H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$ .

Our main result is then the following:

**Theorem 1.1.** *Assume that (1.16) holds. Then, (1.1) is stabilizable in finite time  $L$  if, and only if, (1.1) is exactly controllable at time  $L$ .*

Note that the necessary part is clear from [Remark 2](#) and [Remark 1](#).

Thus, we see that we have to study the controllability of (1.1) at the optimal time of control  $T = L$  (we recall that, in the case  $g = 0$ , (1.1) is exactly controllable at time  $T$  if, and only if,  $T \geq L$ ). We will show that this property is characterized by the criterion of Fattorini in the particular case

$$g(x, y) = g(x), \quad g \in L^2(0, L). \tag{1.17}$$

Indeed, the second result of this paper is

**Theorem 1.2.** *Assume that (1.17) holds. Then, (1.1) is exactly controllable at time  $L$  if, and only if,*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{1.18}$$

**Remark 3.** In fact, (1.18) is a general necessary condition for the approximate controllability. Let us recall that we say that (1.1) is approximately controllable at time  $T$  if, for every  $\epsilon > 0$ , for every  $u^0, u^1 \in L^2(0, L)$ , there exists  $U \in L^2(0, T)$  such that the corresponding solution  $u$  to (1.1) satisfies

$$\|u(T) - u^1\|_{L^2(0, L)} \leq \epsilon.$$

Clearly, it is a weaker property than exact controllability. Let us also recall that this property is equivalent to the following dual one (see e.g. [\[4, Theorem 2.43\]](#)):

$$\forall z \in L^2(0, L), \quad \left( B^*S(t)^*z = 0 \text{ for a.e. } t \in (0, T) \right) \implies z = 0. \tag{1.19}$$

Thus, we see that (1.18) is nothing but the property (1.19) only for  $z \in \ker(\lambda - A^*)$  since  $S(t)^*z = e^{\lambda t}z$  for  $z \in \ker(\lambda - A^*)$ . This condition (1.18) is misleadingly known as the Hautus test [\[9\]](#) in finite dimension, despite it has been introduced earlier by H.O. Fattorini in [\[7\]](#) and in a much larger setting. Finally, let us mention that it has also been proved in [\[1\]](#) that (1.18) characterizes the exponential stabilization of parabolic systems.

**Remark 4.** We will exhibit functions  $g$  such that (1.18) does not hold for an arbitrary large number of  $\lambda$ , see Remark 7 below. On the other hand, we can check that (1.18) is satisfied for any  $g \in L^2((0, L) \times (0, L))$  satisfying one of the following conditions:

- i)  $A^*$  has no eigenvalue (as it is the case when  $g = 0$ ).
- ii)  $g$  is small enough:  $\|g\|_{L^2} < \frac{\sqrt{2}}{L}$ .
- iii)  $g$  is of Volterra type (that is it satisfies (1.3)).

The point ii) follows from the invertibility of transformations  $\text{Id} - Q$  for  $\|Q\|_{\mathcal{L}(L^2)} < 1$ . The point iii) follows from the invertibility of Volterra transformations.

Let us notice that we can also consider equations of the more general form

$$\left\{ \begin{array}{l} \tilde{u}_t(t, x) - \tilde{u}_x(t, x) \\ \quad = \int_0^L \tilde{g}(x, y)\tilde{u}(t, y) dy + f(x)\tilde{u}(t, 0) + d(x)\tilde{u}(t, x), \quad t \in (0, T), x \in (0, L), \\ \tilde{u}(t, L) = \int_0^L \tilde{u}(t, y)\gamma(y) dy + \tilde{U}(t), \quad t \in (0, T), \\ \tilde{u}(0, x) = \tilde{u}^0(x), \quad x \in (0, L), \end{array} \right.$$

where  $f, d, \gamma : (0, L) \rightarrow \mathbb{C}$  and  $\tilde{g} : (0, L) \times (0, L) \rightarrow \mathbb{C}$  are regular enough. Performing a transformation of Volterra type, it can actually be reduced to an equation like (1.1).

Let us conclude the introduction by pointing out that Theorem 1.1 still holds if we consider states and controls taking their values into  $\mathbb{R}$  instead of  $\mathbb{C}$  provided that

$$g(x, y) \in \mathbb{R} \text{ for a.e. } (x, y) \in (0, L) \times (0, L). \tag{1.20}$$

This follows from the fact that, if (1.20) holds and if the control system (1.1), with real valued states and controls, is exactly controllable at time  $L$ , then the functions  $k$  and  $U$  constructed in the proof of Proposition 2.4 below are real valued functions. Concerning Theorem 1.2, it also still holds for real valued states and controls if  $g$  is real valued (but, of course, we still have to consider in (1.18) complex valued functions and complex  $\lambda$ ).

## 2. Finite-time stabilization

### 2.1. Presentation of the method

Let us write  $A = A_0 + G$  where the unbounded linear operator  $A_0 : D(A_0) \subset L^2(0, L) \rightarrow L^2(0, L)$  is defined by

$$A_0u = u_x, \quad D(A_0) = D(A),$$



and the bounded linear operator  $G : L^2(0, L) \rightarrow L^2(0, L)$  is defined by

$$Gu = \int_0^L g(\cdot, y)u(y) dy.$$

Note that the adjoint  $A_0^* : D(A_0^*) \subset L^2(0, L) \rightarrow L^2(0, L)$  of  $A_0$  is the operator

$$A_0^*z = -z_x, \quad D(A_0^*) = \{z \in H^1(0, L) \mid z(0) = 0\}.$$

We first perform some formal computations to explain the ideas of our method. We recall that the strategy is to map the initial equation

$$\begin{cases} \frac{d}{dt}u = (A + B\Gamma)u, & t \in (0, +\infty), \\ u(0) = u^0, \end{cases} \tag{2.1}$$

into the finite-time stable target equation

$$\begin{cases} \frac{d}{dt}w = A_0w, & t \in (0, +\infty), \\ w(0) = w^0, \end{cases} \tag{2.2}$$

for some operator  $\Gamma$  and by means of a transformation  $P$  (independent of the time  $t$ ):

$$u = Pw.$$

If  $u = Pw$  where  $w$  solves (2.2), then

$$\frac{d}{dt}u = \frac{d}{dt}(Pw) = P\left(\frac{d}{dt}w\right) = PA_0w, \tag{2.3}$$

and

$$(A + B\Gamma)u = (AP + B\Gamma P)w. \tag{2.4}$$

As a result,  $u$  solves (2.1) if the right-hand sides of (2.3) and (2.4) are equals, that is, if  $P$  and  $\Gamma$  satisfy

$$PA_0 = AP + B\Gamma P.$$

Taking the adjoints, this is equivalent to

$$A_0^*P^* = P^*A^* + P^*\Gamma^*B^*. \tag{2.5}$$

By (2.5), we mean that

$$P^*(D(A^*)) \subset D(A_0^*) = D(A^*), \tag{2.6}$$

$$A_0^*P^*z = P^*A^*z + P^*\Gamma^*B^*z, \quad \forall z \in D(A^*). \tag{2.7}$$

The following proposition gives the rigorous statement of what we have just discussed (the proof is given in [Appendix B](#)).

**Proposition 2.1.** *Assume that there exist a bounded linear operator  $P : L^2(0, L) \rightarrow L^2(0, L)$  and a bounded linear form  $\Gamma : L^2(0, L) \rightarrow \mathbb{C}$  such that:*

- i) (2.6)–(2.7) hold.*
- ii)  $P$  is invertible.*

*Then, for every  $u^0 \in L^2(0, L)$ , if  $w \in C^0([0, +\infty); L^2(0, L))$  denotes the solution to (2.2) with  $w^0 = P^{-1}u^0$ , then  $u = Pw$  is the solution to (1.13) and it satisfies (1.14) for  $T = L$ .*

Let us now “split” the equation (2.7). We recall that  $D(A^*)$  is a Hilbert space and  $B^*$  is continuous for the norm of  $D(A^*)$  (see the introduction). Thus, its kernel  $\ker B^*$  is closed for this norm and we can write the orthogonal decomposition

$$D(A^*) = \ker B^* \oplus (\ker B^*)^\perp,$$

where  $V^\perp$  denotes the orthogonal of a subspace  $V$  in  $D(A^*)$ . Noting that  $B^*$  is a bijection from  $(\ker B^*)^\perp$  to  $\mathbb{C}$  (with inverse denoted by  $(B^*)^{-1}$ ), we see that (2.7) holds if, and only if,

$$A_0^*P^*z - P^*A^*z = 0, \quad \forall z \in \ker B^*, \tag{2.8}$$

and

$$P^*\Gamma^* = (A_0^*P^* - P^*A^*)(B^*)^{-1}. \tag{2.9}$$

It follows from this observation that it is enough to establish the existence of  $P$  such that (2.8) holds and  $P$  is invertible. The map  $\Gamma$  will then be defined as the adjoint of the linear map  $\Psi : \mathbb{C} \rightarrow L^2(0, L)$  defined by

$$\Psi = ((P^*)^{-1}A_0^*P^* - A^*)(B^*)^{-1}. \tag{2.10}$$

Note that  $P^* : D(A^*) \rightarrow D(A^*)$  is continuous by the closed graph theorem, so that  $\Psi$  defined by (2.10) is bounded.

Let us summarize the discussion:

**Proposition 2.2.** *Let  $P : L^2(0, L) \rightarrow L^2(0, L)$  be a bounded linear operator such that (2.6) holds and  $P$  is invertible. Then, there exists a bounded linear form  $\Gamma : L^2(0, L) \rightarrow \mathbb{C}$  such that (2.7) holds if, and only if,  $P^*$  satisfies (2.8).*

A discussion on other expressions of  $\Gamma$  than (2.10) is given in Section 2.4 below.

2.2. Construction of the transformation

In this section, we are going to construct a map  $P$  such that (2.6) and (2.8) hold. We look for  $P$  in the form

$$P = \text{Id} - K, \tag{2.11}$$

where  $K : L^2(0, L) \rightarrow L^2(0, L)$  is an integral operator defined by

$$Kz(x) = \int_0^L k(x, y)z(y) dy,$$

with  $k \in L^2((0, L) \times (0, L))$ . Clearly, its adjoint is

$$K^*z(x) = \int_0^L k^*(x, y)z(y) dy,$$

where we set

$$k^*(x, y) = \overline{k(y, x)}.$$

Let us recall that  $K$ , as well as  $K^*$ , is compact on  $L^2(0, L)$ .

For the expression (2.11), (2.6) now read as

$$K^*(D(A^*)) \subset D(A^*), \tag{2.12}$$

and (2.8) becomes

$$-A_0^*K^*z + K^*A_0z + K^*G^*z - G^*z = 0, \quad \forall z \in \ker B^*. \tag{2.13}$$

Let us now translate these properties in terms of the kernel  $k^*$ .

**Proposition 2.3.** *Assume that*

$$k^* \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+), \tag{2.14}$$

and let  $k_+^* \in L^2(\partial\mathcal{T}_+)$  be the trace on  $\mathcal{T}_+$  of the restriction of  $k^*$  to  $\mathcal{T}_+$ . Then,

i) (2.12) holds if, and only if,

$$k_+^*(0, y) = 0, \quad y \in (0, L). \tag{2.15}$$

ii) (2.13) holds if, and only if,

$$k_x^*(x, y) + k_y^*(x, y) + \int_0^L \overline{g(y, \sigma)} k^*(x, \sigma) d\sigma - \overline{g(y, x)} = 0, \quad x, y \in (0, L). \quad (2.16)$$

Observe that if  $k^* \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$ , then  $k_x^*, k_y^* \in L^2((0, L) \times (0, L))$  and (2.16) is understood as an equality for almost every  $(x, y) \in (0, L) \times (0, L)$ .

**Proof.** Let us first prove the equivalence between (2.12) and (2.15). Since  $k^* \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$ , we have  $K^*z \in H^1(0, L)$  for every  $z \in L^2(0, L)$  with (see Proposition A.2 ii))

$$K^*z(0) = \int_0^L k_+^*(0, y)z(y) dy.$$

Thus, (2.12) holds if, and only if,  $K^*z(0) = 0$  for every  $z \in D(A^*)$ , which gives (2.15) by density of  $D(A^*)$  in  $L^2(0, L)$ .

Let us now establish the equivalence between (2.13) and (2.16). Let us compute each terms in the left-hand side of (2.13) for any  $z \in D(A^*)$ . For the first term we have (see Proposition A.2 ii))

$$\begin{aligned} -A_0^*K^*z(x) &= \partial_x \left( \int_0^L k^*(\cdot, y)z(y) dy \right) (x) \\ &= (k_-^*(x, x) - k_+^*(x, x))z(x) + \int_0^L k_x^*(x, y)z(y) dy, \end{aligned}$$

where  $k_-^* \in L^2(\partial\mathcal{T}_-)$  (resp.  $k_+^* \in L^2(\partial\mathcal{T}_+)$ ) denotes the trace on  $\mathcal{T}_-$  (resp.  $\mathcal{T}_+$ ) of the restriction of  $k^*$  to  $\mathcal{T}_-$  (resp.  $\mathcal{T}_+$ ). On the other hand, integrating by parts the second term and using  $z(0) = 0$  (since  $z \in D(A^*)$ ), we have (see Proposition A.2 i))

$$\begin{aligned} K^*A_0^*z(x) &= - \int_0^L k^*(x, y)z'(y) dy \\ &= \int_0^L k_y^*(x, y)z(y) dy - (k_-^*(x, x) - k_+^*(x, x))z(x) - k_+^*(x, L)z(L). \end{aligned}$$

Finally, the remaining term gives

$$K^*G^*z(x) - G^*z(x) = \int_0^L k^*(x, y) \left( \int_0^L \overline{g(\sigma, y)}z(\sigma) d\sigma \right) dy - \int_0^L \overline{g(y, x)}z(y) dy$$

$$= \int_0^L \left( \int_0^L k^*(x, \sigma) \overline{g(y, \sigma)} d\sigma - \overline{g(y, x)} \right) z(y) dy.$$

As a result, summing all the previous equalities, we have

$$\begin{aligned} & -A_0^* K^* z(x) + K^* A_0^* z(x) + K^* G^* z(x) - G^* z(x) \\ &= \int_0^L \left( k_x^*(x, y) + k_y^*(x, y) + \int_0^L k^*(x, \sigma) \overline{g(y, \sigma)} d\sigma - \overline{g(y, x)} \right) z(y) dy - k_+^*(x, L) z(L), \end{aligned} \tag{2.17}$$

for every  $z \in D(A^*)$ . In particular, we obtain that (2.13) is equivalent to

$$\int_0^L \left( k_x^*(x, y) + k_y^*(x, y) + \int_0^L k^*(x, \sigma) \overline{g(y, \sigma)} d\sigma - \overline{g(y, x)} \right) z(y) dy = 0,$$

for every  $z \in \ker B^* = H_0^1(0, L)$ . Since  $H_0^1(0, L)$  is dense in  $L^2(0, L)$ , this is equivalent to the equation (2.16). □

**Remark 5.** In the first step of the proof we have in fact establish that (2.15) is equivalent to

$$K^* (L^2(0, L)) \subset D(A^*). \tag{2.18}$$

We see that the operator  $K^*$  has a regularizing effect (under assumption (2.14)).

*2.2.1. Existence of the kernel*

Viewing  $x$  as the time parameter in (2.15)–(2.16), it is clear that these equations have at least one solution  $k^* \in C^0([0, L]; L^2(0, L))$ , if we add any artificial  $L^2$  boundary condition at  $(x, 0)$ . In this section, we fix a particular boundary condition such that  $k^*$  satisfies, in addition, the final condition

$$k^*(L, y) = 0, \quad y \in (0, L). \tag{2.19}$$

This property will be used to establish the invertibility of the Fredholm transformation associated with this  $k^*$ , see Section 2.3 below.

**Proposition 2.4.** *Assume that (1.1) is exactly controllable at time  $L$ . Then, there exists  $U \in L^2(0, L)$  such that the solution  $k^* \in C^0([0, L]; L^2(0, L))$  to*

$$\begin{cases} k_x^*(x, y) + k_y^*(x, y) + \int_0^L \overline{g(y, \sigma)} k^*(x, \sigma) d\sigma - \overline{g(y, x)} = 0, & x, y \in (0, L), \\ k^*(x, L) = U(x), & x \in (0, L), \\ k^*(L, y) = 0, & y \in (0, L), \end{cases} \tag{2.20}$$

satisfies

$$k^*(0, y) = 0, \quad y \in (0, L). \tag{2.21}$$

**Proof.** Since  $x$  plays the role of the time, let us introduce

$$\tilde{k}(t, y) = \overline{k^*(L - t, y)}.$$

Thus, we want to prove that there exists  $\tilde{U} \in L^2(0, L)$  such that the corresponding solution  $\tilde{k} \in C^0([0, L]; L^2(0, L))$  to

$$\begin{cases} \tilde{k}_t(t, y) - \tilde{k}_y(t, y) = \int_0^L g(y, \sigma) \tilde{k}(t, \sigma) d\sigma - g(y, L - t), & t, y \in (0, L), \\ \tilde{k}(t, L) = \tilde{U}(t), & t \in (0, L), \\ \tilde{k}(0, y) = 0, & y \in (0, L), \end{cases} \tag{2.22}$$

satisfies

$$\tilde{k}(L, y) = 0, \quad y \in (0, L). \tag{2.23}$$

This is a control problem, which has a solution by assumption. Indeed, let  $p \in C^0([0, L]; L^2(0, L))$  be the free solution to the nonhomogeneous equation

$$\begin{cases} p_t(t, y) - p_y(t, y) = \int_0^L g(y, \sigma) p(t, \sigma) d\sigma - g(y, L - t), & t, y \in (0, L), \\ p(t, L) = 0, & t \in (0, L), \\ p(0, y) = 0, & y \in (0, L), \end{cases}$$

and let  $q \in C^0([0, L]; L^2(0, L))$  be the controlled solution going from 0 to  $-p(L, \cdot)$ :

$$\begin{cases} q_t(t, y) - q_y(t, y) = \int_0^L g(y, \sigma) q(t, \sigma) d\sigma, & t, y \in (0, L), \\ q(t, L) = \tilde{U}(t), & t \in (0, L), \\ q(0, y) = 0, \quad q(L, y) = -p(L, y), & y \in (0, L). \end{cases}$$

Then, the function  $\tilde{k} \in C^0([0, L]; L^2(0, L))$  defined by

$$\tilde{k} = p + q,$$

satisfies (2.22)–(2.23).  $\square$

2.2.2. Regularity of the kernel

The next step is to establish the regularity (2.14) for  $k^*$  provided by Proposition 2.4.

**Proposition 2.5.** *Let  $U \in L^2(0, L)$  and let  $k^* \in C^0([0, L]; L^2(0, L))$  be the corresponding solution to (2.20). If  $k^*$  satisfies (2.21) and (1.16) holds, then*

$$U \in H^1(0, L), \quad k^* \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+).$$

The proof of Proposition 2.5 relies on the following lemma:

**Lemma 2.1.** *Let  $f \in L^2((0, L) \times (0, L))$ ,  $V \in L^2(0, L)$  and  $v^0 \in L^2(0, L)$ .*

i) *The unique solution  $v \in C^0([0, L]; L^2(0, L))$  to*

$$\begin{cases} v_x(x, y) + v_y(x, y) = f(x, y), & x, y \in (0, L), \\ v(x, L) = V(x), & x \in (0, L), \\ v(L, y) = v^0(y), & y \in (0, L), \end{cases} \tag{2.24}$$

is given by

$$v(x, y) = \begin{cases} V(L + x - y) - \int_x^{L+x-y} f(s, s + y - x) ds, & \text{if } (x, y) \in \mathcal{T}_+, \\ v^0(L + y - x) - \int_x^L f(s, s + y - x) ds, & \text{if } (x, y) \in \mathcal{T}_-. \end{cases}$$

ii) *If  $V \in H^1(0, L)$  (resp.  $v^0 \in H^1(0, L)$ ) and  $f_y \in L^2(\mathcal{T}_+)$  (resp.  $f_y \in L^2(\mathcal{T}_-)$ ), then  $v \in H^1(\mathcal{T}_+)$  (resp.  $v \in H^1(\mathcal{T}_-)$ ).*

iii) *If  $f_y \in L^2(\mathcal{T}_+)$  and  $v(0, \cdot) \in H^1(0, L)$ , then  $V \in H^1(0, L)$ .*

**Proof.** Let us apply Lemma 2.1 with  $V = U \in L^2(0, L)$ ,  $v^0 = 0$  and

$$f(x, y) = f_1(x, y) + f_2(x, y),$$

$$f_1(x, y) = - \int_0^L \overline{g(y, \sigma)} k^*(x, \sigma) d\sigma, \quad f_2(x, y) = \overline{g(y, x)}.$$

Since  $k^*, g \in L^2((0, L) \times (0, L))$ , we have  $f_1, f_2 \in L^2((0, L) \times (0, L))$ . By uniqueness, the corresponding solution  $v$  to (2.24) is equal to  $k^*$ . Since  $g \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$  by assumption (1.16), we have  $(f_2)_y \in L^2(\mathcal{T}_+)$  and  $(f_2)_y \in L^2(\mathcal{T}_-)$  by definition. On the other hand, for a.e.  $x \in (0, L)$ , the map  $f_1(x) : y \mapsto f_1(x, y)$  belongs to  $H^1(0, L)$  with derivative (see Proposition A.2 ii))

$$f_1(x)'(y) = - \int_0^L \overline{g_x(y, \sigma)} k^*(x, \sigma) d\sigma - (g_-(y, y) - g_+(y, y)) k^*(x, y).$$

This shows that  $(f_1)_y \in L^2(\mathcal{T}_-)$  and  $(f_1)_y \in L^2(\mathcal{T}_+)$  (see Proposition A.1). Finally, since  $k^*$  satisfies  $k^*(0, y) = 0$  for a.e.  $y \in (0, L)$ , by Lemma 2.1 iii) we have  $U \in H^1(0, L)$ . Then, it follows from Lemma 2.1 ii) that  $k^* \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$ . □

### 2.3. Invertibility of the transformation

To conclude the whole proof of Theorem 1.1, it only remains to establish the invertibility of the transformation  $\text{Id} - K^*$  with  $k^*$  provided by Proposition 2.4. Let us start with a general lemma on the injectivity of maps  $P^*$  for  $P$  satisfying (2.6)–(2.8).

**Lemma 2.2.** *Let  $P : L^2(0, L) \rightarrow L^2(0, L)$  be a bounded linear operator such that (2.6)–(2.8) hold. Then, we have*

$$\ker P^* = \{0\},$$

if, and only if, the following four conditions hold:

- i)  $\ker P^* \subset D(A^*)$ .
- ii)  $\ker P^* \subset \ker B^*$ .
- iii)  $\dim \ker P^* < +\infty$ .
- iv)  $\ker(\lambda - A^*) \cap \ker B^* = \{0\}$  for every  $\lambda \in \mathbb{C}$ .

**Proof.** Let us denote

$$N = \ker P^*.$$

Assume first that i), ii), iii) and iv) hold. We want to prove that  $N = \{0\}$ . We argue by contradiction: assume that  $N \neq \{0\}$ . Let us prove that  $N$  is stable by  $A^*$ . By i) we have  $N \subset D(A^*)$ . Let then  $z \in N$  and let us show that  $A^*z \in N$ . Since  $N \subset \ker B^*$  by ii), we can apply (2.8) to  $z$  and obtain

$$P^* A^* z = A_0^* P^* z.$$



Since  $z \in \ker P^*$  by definition, this gives

$$P^*A^*z = 0,$$

and shows that  $A^*z \in \ker P^* = N$ . Consequently, the restriction  $A^*|_N$  of  $A^*$  to  $N$  is a linear operator from  $N$  to  $N$ . Since  $N$  is finite dimensional by iii) and  $N \neq \{0\}$ ,  $A^*|_N$  has at least one eigenvalue  $\lambda \in \mathbb{C}$ . Let  $\xi \in N$  be a corresponding eigenfunction. Thus,

$$\xi \in \ker(\lambda - A^*) \cap \ker B^*,$$

but

$$\xi \neq 0,$$

which is a contradiction with iv). As a result, we must have  $N = \{0\}$ .

Conversely, assume now that  $\ker P^* = \{0\}$ . It is clear that i), ii) and iii) hold. Let  $\lambda \in \mathbb{C}$  and  $z \in \ker(\lambda - A^*) \cap \ker B^*$ . We want to prove that  $z = 0$ . By (2.8), we have

$$A_0^*P^*z = \lambda P^*z,$$

that is

$$(\lambda - A_0^*)P^*z = 0.$$

Since  $\lambda - A_0^*$  (with domain  $D(A_0^*)$ ) is injective and so is  $P^*$  by assumption, this gives  $z = 0$ .  $\square$

**Proposition 2.6.** *Assume that (1.1) is exactly controllable at time  $L$  and that (1.16) holds. Then, the map  $\text{Id} - K^*$ , with  $k^*$  provided by Proposition 2.4, is invertible.*

**Proof.** Since  $K^*$  is a compact operator, by Fredholm alternative it is equivalent to prove that  $\text{Id} - K^*$  is injective. In addition, the Fredholm alternative also gives

$$\dim \ker(\text{Id} - K^*) < +\infty.$$

Since  $\text{Id} - K^*$  satisfies (2.12)–(2.13), by Lemma 2.2 it is then equivalent to establish that

$$\ker(\text{Id} - K^*) \subset D(A^*), \quad \ker(\text{Id} - K^*) \subset \ker B^*.$$

The first inclusion follows from Remark 5 and the second inclusion follows from the fact that

$$B^*K^*z = 0, \quad \forall z \in L^2(0, L),$$

which is equivalent to the condition (2.19).  $\square$

2.4. *Feedback control law*

The proof of [Theorem 1.1](#) is by now complete but we want to give a more explicit formula for  $\Gamma$ . We recall that its adjoint  $\Gamma^*$  is given by (see [\(2.10\)](#))

$$\Gamma^* = (P^*)^{-1} (A_0^* P^* - P^* A^*) (B^*)^{-1}.$$

Actually, we already computed  $A_0^* P^* z - P^* A^* z$  for any  $z \in D(A^*)$  in [\(2.17\)](#) and we obtained that

$$A_0^* P^* z - P^* A^* z = -k_+^*(\cdot, L)z(L).$$

Thus,

$$P^* \Gamma^* a = -k_+^*(\cdot, L)a, \quad a \in \mathbb{C}.$$

Computing the adjoints, we obtain

$$\Gamma u = - \int_0^L k_-(L, x) P^{-1} u(x) dx, \quad u \in L^2(0, L).$$

It is interesting to see that the open loop control  $U$  provided by [Proposition 2.4](#) defines the closed loop control  $\Gamma$  (since  $k_-(L, x) = \overline{U(x)}$  for a.e.  $x \in (0, L)$ ).

Let us now recall that  $P$  is of the form  $P = \text{Id} - K$  and that the inverse of such an operator is also of the form  $\text{Id} - H$  (with  $H = -(\text{Id} - K)^{-1}K$ ). Moreover, since  $K$  is an integral operator so is  $H$ , with kernel  $h(\cdot, y) = -(\text{Id} - K)^{-1}k(\cdot, y)$ . We can check that  $h$  inherits the regularity of  $k$  and satisfies a similar equation:

$$\begin{cases} h_x(x, y) + h_y(x, y) - \int_0^L g(\sigma, y)h(x, \sigma)d\sigma + g(x, y) = 0, & x, y \in (0, L), \\ h(x, 0) = 0, \quad h(x, L) = 0, & x \in (0, L). \end{cases}$$

Finally, a simple computation shows that  $\Gamma$  is given by

$$\Gamma u = \int_0^L h_-(L, y)u(y) dy,$$

where  $h_- \in L^2(\partial\mathcal{T}_-)$  denotes the trace on  $\mathcal{T}_-$  of the restriction of  $h$  to  $\mathcal{T}_-$ .

### 3. Controllability

The aim of this section is to study the controllability properties of (1.1) at the optimal time  $T = L$  to provide easily checkable conditions to apply Theorem 1.1. Let us first mention that the controllability of one-dimensional systems generated by  $C_0$ -groups has already been investigated in a series of papers [12] and [8]. However, all these papers do not really focus on the optimal time of controllability, which is crucial to apply our stabilization theorem. Let us also point out that the method developed in [15] seems ineffective because of the integral term  $\int_x^L g(x, y)u(t, y) dy$  in (1.1). Finally, let us mention the result [17, Theorem 2.6] for the distributed controllability of compactly perturbed systems (the case of the optimal time can not be treated though).

In order to have a good spectral theory, we consider system (1.1) with periodic boundary conditions:

$$\begin{cases} \tilde{u}_t(t, x) - \tilde{u}_x(t, x) = \int_0^L g(x, y)\tilde{u}(t, y) dy, & t \in (0, T), x \in (0, L), \\ \tilde{u}(t, L) - \tilde{u}(t, 0) = \tilde{U}(t), & t \in (0, T), \\ \tilde{u}(0, x) = \tilde{u}^0(x), & x \in (0, L), \end{cases} \tag{3.1}$$

where  $\tilde{u}^0 \in L^2(0, L)$  and  $\tilde{U} \in L^2(0, T)$ . In the abstract form, (3.1) reads

$$\begin{cases} \frac{d}{dt} \tilde{u} = \tilde{A}\tilde{u} + \tilde{B}\tilde{U}, & t \in (0, T), \\ \tilde{u}(0) = \tilde{u}^0, \end{cases}$$

where  $\tilde{A}$  is the operator  $A$  (see (1.7)) but now with domain

$$D(\tilde{A}) = \{ \tilde{u} \in H^1(0, L) \mid \tilde{u}(L) = \tilde{u}(0) \},$$

and  $\tilde{B}$  is the operator  $B$  (see (1.10)) but now considered as an operator of  $\mathcal{L}(\mathbb{C}, D(\tilde{A}^*))'$ . The adjoints of these operators also remain unchanged (see (1.8) and (1.11)), except for their domain:

$$D(\tilde{A}^*) = D(\tilde{A}), \quad \tilde{B}^* \in \mathcal{L}(D(\tilde{A}^*), \mathbb{C}).$$

We can check that  $\tilde{A}$  generates a  $C^0$ -group  $(\tilde{S}(t))_{t \in \mathbb{R}}$  and  $\tilde{B}$  is admissible. Thus, (3.1) is well-posed, that is, for every  $\tilde{u}^0 \in L^2(0, L)$  and every  $\tilde{U} \in L^2(0, T)$ , there exists a unique solution  $\tilde{u} \in C^0([0, T]; L^2(0, L))$  to (3.1) and, in addition, there exists  $C > 0$  (which does not depend on  $\tilde{u}^0$  nor  $\tilde{U}$ ) such that

$$\|\tilde{u}\|_{C^0([0, T]; L^2(0, L))} \leq C \left( \|\tilde{u}^0\|_{L^2(0, L)} + \|\tilde{U}\|_{L^2(0, T)} \right). \tag{3.2}$$

The following proposition shows that it is indeed equivalent to consider (3.1) or (1.1) from a controllability point of view.

**Proposition 3.1.** *(1.1) is exactly controllable at time  $T$  if, and only if, (3.1) is exactly controllable at time  $T$ .*

Roughly speaking, to prove Proposition 3.1, it suffices to take  $\tilde{u}^0 = u^0$  and  $U(t) = \tilde{u}(t, 0) + \tilde{U}(t)$ . We postpone the rigorous proof to Appendix C.

In addition, note that

$$\ker(\lambda - A^*) \cap \ker B^* = \ker(\lambda - \tilde{A}^*) \cap \ker \tilde{B}^*,$$

for every  $\lambda \in \mathbb{C}$ . As a result, (1.18) is equivalent to

$$\ker(\lambda - \tilde{A}^*) \cap \ker \tilde{B}^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{3.3}$$

### 3.1. Bases and problem of moments in Hilbert spaces

Let us recall here some basic facts about bases and the problem of moments in Hilbert spaces. We follow the excellent textbook [21]. Let  $H$  be a complex Hilbert space. We say that  $\{f_k\}_{k \in \mathbb{Z}}$  is a basis in  $H$  if, for every  $f \in H$  there exists a unique sequence of scalar  $\{\alpha_k\}_{k \in \mathbb{Z}}$  such that  $f = \sum_{k \in \mathbb{Z}} \alpha_k f_k$ . We say that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Riesz basis in  $H$  if it is the image of an orthonormal basis of  $H$  through an isomorphism. We can prove that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Riesz basis if, and only if,  $\{f_k\}_{k \in \mathbb{Z}}$  is complete in  $H$  and there exist  $m, M > 0$  such that, for every  $N \in \mathbb{N}$ , for every scalars  $\alpha_{-N}, \dots, \alpha_N$ , we have

$$m \sum_{k=-N}^N |\alpha_k|^2 \leq \left\| \sum_{k=-N}^N \alpha_k f_k \right\|_H^2 \leq M \sum_{k=-N}^N |\alpha_k|^2. \tag{3.4}$$

See e.g. [21, Chapter 1, Theorem 9].

A useful criterion to prove that a sequence is a Riesz basis is the theorem of Bari (see e.g. [21, Chapter 1, Theorem 15]). It states that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Riesz basis of  $H$  if  $\{f_k\}_{k \in \mathbb{Z}}$  is  $\omega$ -independent, that is, for every sequence of scalars  $\{c_k\}_{k \in \mathbb{Z}}$ ,

$$\sum_{k \in \mathbb{Z}} c_k f_k = 0 \implies (c_k = 0, \quad \forall k \in \mathbb{Z}), \tag{3.5}$$

and if  $\{f_k\}_{k \in \mathbb{Z}}$  is quadratically close to some orthonormal basis  $\{e_k\}_{k \in \mathbb{Z}}$  of  $H$ , that is

$$\sum_{k \in \mathbb{Z}} \|f_k - e_k\|_H^2 < +\infty.$$

On the other hand, we say that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Bessel sequence in  $H$  if, for every  $f \in H$ , we have

$$\sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle_H|^2 < +\infty.$$

We can prove that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Bessel sequence in  $H$  if, and only if,  $\{f_k\}_{k \in \mathbb{Z}}$  satisfies the second inequality in (3.4). See e.g. [21, Chapter 2, Theorem 3].

Finally, we say that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Riesz–Fischer sequence in  $H$  if, for every sequence of scalars  $\{c_k\}_{k \in \mathbb{Z}}$  that belongs to  $\ell^2(\mathbb{Z})$ , there exists (at least) a solution  $f \in H$  to the problem of moments

$$c_k = \langle f, f_k \rangle_H, \quad \forall k \in \mathbb{Z}.$$

We can prove that  $\{f_k\}_{k \in \mathbb{Z}}$  is a Riesz–Fischer sequence in  $H$  if, and only if,  $\{f_k\}_{k \in \mathbb{Z}}$  satisfies the first inequality in (3.4). See e.g. [21, Chapter 2, Theorem 3].

Observe then that, a Riesz basis is nothing but a complete Bessel and Riesz–Fischer sequence. We refer to [21, Chapter 4] for more details on the problem of moments.

To prove Theorem 1.2, the idea is to write the controllability problem as a problem of moments. To achieve this goal, and to prove that the resulting problem of moments indeed has a solution, we first need to establish some spectral properties of our operator  $\tilde{A}^*$ .

### 3.2. Spectral properties of $\tilde{A}^*$

From now on, we assume that  $g$  depends only on its first variable  $x$ :

$$g(x, y) = g(x), \quad g \in L^2(0, L). \tag{3.6}$$

The first proposition gives the basic spectral properties of  $\tilde{A}^*$ .

**Proposition 3.2.** *Assume that (3.6) holds. Then,*

*i) For every  $\lambda \in \mathbb{C}$ , we have*

$$\ker(\lambda - \tilde{A}^*) = \left\{ ae^{-\lambda x} + bw_\lambda(x) \mid (a, b) \in \mathbb{C}^2, \quad H(\lambda) \begin{pmatrix} a \\ b \end{pmatrix} = 0 \right\},$$

*where we have introduced the matrix*

$$H(\lambda) = \begin{pmatrix} 1 - e^{-\lambda L} & -w_\lambda(L) \\ \int_0^L \overline{g(x)} e^{-\lambda x} dx & \int_0^L \overline{g(x)} w_\lambda(x) dx - 1 \end{pmatrix},$$

*and the function*

$$w_\lambda(x) = \int_0^x e^{-\lambda(x-\sigma)} d\sigma = \begin{cases} \frac{1 - e^{-\lambda x}}{\lambda} & \text{if } \lambda \neq 0, \\ x & \text{if } \lambda = 0. \end{cases}$$

ii) We have

$$\sigma(\tilde{A}^*) = \left\{ \lambda_k = \frac{2ik\pi}{L} \mid k \in \mathbb{Z}, k \neq 0 \right\} \cup \left\{ \lambda_0 = \int_0^L \overline{g(x)} dx \right\}.$$

**Proof.** Let us prove i). Let  $\lambda \in \mathbb{C}$ . Let  $z \in \ker(\lambda - \tilde{A}^*)$ , that is,

$$\begin{cases} z \in H^1(0, L), & z(L) = z(0), \\ \lambda z(x) + z'(x) - \int_0^L \overline{g(\sigma)} z(\sigma) d\sigma = 0, & x \in (0, L). \end{cases} \tag{3.7}$$

Solving the ODE in (3.7) yields

$$z(x) = e^{-\lambda x} z(0) + w_\lambda(x) I, \tag{3.8}$$

with

$$I = \int_0^L \overline{g(\sigma)} z(\sigma) d\sigma.$$

From the boundary condition  $z(L) = z(0)$  we obtain the relation

$$(1 - e^{-\lambda L}) z(0) - w_\lambda(L) I = 0.$$

To obtain a second relation, we multiply (3.8) by  $\bar{g}$  and integrate over  $(0, L)$ , so that

$$\left( \int_0^L \overline{g(x)} e^{-\lambda x} dx \right) z(0) + \left( \int_0^L \overline{g(x)} w_\lambda(x) dx - 1 \right) I = 0.$$

Conversely, let

$$z(x) = a e^{-\lambda x} + b w_\lambda(x),$$

where  $(a, b) \in \mathbb{C}^2$  is such that

$$H(\lambda) \begin{pmatrix} a \\ b \end{pmatrix} = 0. \tag{3.9}$$

Clearly,  $z \in H^1(0, L)$ . From the first equation of (3.9) and  $w_\lambda(0) = 0$ , we have  $z(L) = z(0)$ . From the second equation of (3.9),  $z$  solves the ODE in (3.7).

Let us now turn out to the proof of ii). The map

$$\begin{aligned} \ker H(\lambda) &\longrightarrow \ker(\lambda - \tilde{A}^*) \\ \begin{pmatrix} a \\ b \end{pmatrix} &\longmapsto ae^{-\lambda x} + bw_\lambda(x) \end{aligned}$$

is an isomorphism (the injectivity can be seen using  $w_\lambda(0) = 0$ ). As a result,

$$\dim \ker(\lambda - \tilde{A}^*) = \dim \ker H(\lambda), \quad \forall \lambda \in \mathbb{C}.$$

In particular,

$$\lambda \in \sigma(\tilde{A}^*) \iff \det H(\lambda) = 0.$$

Let us now compute more precisely  $\det H(\lambda)$ . Observe that

$$1 - e^{-\lambda x} - \lambda w_\lambda(x) = 0, \quad \forall \lambda \in \mathbb{C}, \forall x \in [0, L].$$

Thus, adding  $\lambda$  times the second column of the matrix  $H(\lambda)$  to its first column, we obtain

$$\det H(\lambda) = \det \begin{pmatrix} 0 & -w_\lambda(L) \\ \int_0^L \overline{g(x)} dx - \lambda & \int_0^L \overline{g(x)} w_\lambda(x) dx - 1 \end{pmatrix},$$

so that

$$\det H(\lambda) = w_\lambda(L) \left( \int_0^L \overline{g(x)} dx - \lambda \right).$$

Finally, from the very definition of  $w_\lambda$ , we can check that

$$w_\lambda(L) = 0 \iff \lambda \in \left\{ \frac{2ik\pi}{L} \mid k \in \mathbb{Z}, k \neq 0 \right\}. \quad \square$$

**Remark 6.** In view of the controllability, we shall always assume that

$$\lambda_0 \neq \lambda_k, \quad \forall k \neq 0. \tag{3.10}$$

Indeed, if (3.10) does not hold, then  $\lambda_0$  is an eigenvalue of geometric multiplicity at least two and (3.1) is then impossible to control since the control operator is one-dimensional. This follows from the general inequality

$$\dim \ker(\lambda - \tilde{A}^*) \leq \dim \operatorname{Im} \tilde{B}^*, \quad \forall \lambda \in \mathbb{C},$$

which is a consequence of (3.3) (and we recall that (3.3) is a necessary condition to the controllability, see Remark 3). Note that (3.10) holds in particular if  $g$  is a real-valued function.

Under assumption (3.10) it is not difficult to see that the eigenspaces of  $\tilde{A}^*$  can be rewritten as

$$\ker(\lambda_k - \tilde{A}^*) = \text{Span} \{ \phi_k \},$$

where

$$\phi_0(x) = 1, \quad \phi_k(x) = e^{-\lambda_k x} + \frac{1}{\lambda_k - \lambda_0} \int_0^L \overline{g(x)} e^{-\lambda_k x} dx. \tag{3.11}$$

Let us now write the property (3.3) more explicitly for the case (3.6) (the proof is straightforward thanks to (3.11)).

**Proposition 3.3.** *Assume that (3.6) and (3.10) hold. Then, (3.3) is equivalent to*

$$1 + \frac{1}{\lambda_k - \lambda_0} \int_0^L \overline{g(x)} e^{-\lambda_k x} dx \neq 0, \quad \forall k \neq 0. \tag{3.12}$$

**Remark 7.** Actually, (3.12) has to be checked only for a finite number of  $k$ . Indeed, (3.12) always holds for  $k$  large enough since

$$\frac{1}{\lambda_k - \lambda_0} \int_0^L \overline{g(x)} e^{-\lambda_k x} dx \xrightarrow[k \rightarrow \pm\infty]{} 0. \tag{3.13}$$

On the other hand, there exist functions  $g$  such that (3.12) fails for an arbitrary large number of  $k$ . Indeed, observe that for real-valued function  $g$ , the equality

$$1 + \frac{1}{\lambda_k - \lambda_0} \int_0^L \overline{g(x)} e^{-\lambda_k x} dx = 0$$

is equivalent to (taking real and imaginary parts)

$$\begin{cases} \int_0^L g(x) \cos\left(\frac{2k\pi}{L}x\right) dx = \int_0^L g(x) dx, \\ \int_0^L g(x) \sin\left(\frac{2k\pi}{L}x\right) dx = \frac{2k\pi}{L}. \end{cases}$$



For instance, for any  $a_0 \in \mathbb{R}$  and any  $N \geq 1$ , the function

$$g(x) = a_0 + \frac{2}{L} \sum_{k=1}^N a_0 \cos\left(\frac{2k\pi}{L}x\right) + \frac{2}{L} \sum_{k=1}^N \frac{2k\pi}{L} \sin\left(\frac{2k\pi}{L}x\right)$$

satisfies these equalities for  $k = 1, \dots, N$ .

The next and last proposition provides all the additional spectral properties required to apply the method of moments.

**Proposition 3.4.** *Assume that (3.6) and (3.10) hold. Then:*

- i) The eigenfunctions  $\{\phi_k\}_{k \in \mathbb{Z}}$  of  $\tilde{A}^*$  form a Riesz basis in  $L^2(0, L)$ .*
- ii) If (3.12) holds, then  $\inf_{k \in \mathbb{Z}} |\tilde{B}^* \phi_k| > 0$ .*
- iii) The set of exponentials  $\{e^{-\lambda_k t}\}_{k \in \mathbb{Z}}$  is a Riesz basis in  $L^2(0, L)$ .*

**Proof.** i) We will use the theorem of Bari previously mentioned. Clearly,  $\{\frac{1}{\sqrt{L}}\phi_k\}_{k \in \mathbb{Z}}$  is quadratically close to the orthonormal basis  $\{\frac{1}{\sqrt{L}}e^{-\frac{2ik\pi}{L}x}\}_{k \in \mathbb{Z}}$ . To prove that  $\{\frac{1}{\sqrt{L}}\phi_k\}_{k \in \mathbb{Z}}$  is  $\omega$ -independent, it suffices to take the inner product of the series in (3.5) with each  $e^{-\frac{2ik\pi}{L}x}$ .

ii) From (3.13) we have  $\tilde{B}^* \phi_k \xrightarrow{k \rightarrow \pm\infty} 1$  and by assumption  $\tilde{B}^* \phi_k \neq 0$  for every  $k \in \mathbb{Z}$ .

iii) Again, it suffices to notice that  $\{e^{-\lambda_k t}\}_{k \in \mathbb{Z}}$  is  $\omega$ -independent and quadratically close to  $\{\frac{1}{\sqrt{L}}e^{\frac{2ik\pi}{L}t}\}_{k \in \mathbb{Z}}$ .  $\square$

### 3.3. Proof of Theorem 1.2

Let us first recall the following fundamental relation between the solution to (3.1) and its adjoint state:

$$\langle \tilde{u}(T), z \rangle_{L^2} - \langle \tilde{u}^0, \tilde{S}(T)^* z \rangle_{L^2} = \int_0^T \tilde{U}(t) \overline{\tilde{B}^* \tilde{S}(T-t)^* z} dt, \quad \forall z \in L^2(0, L). \tag{3.14}$$

We have now everything we need to apply the method of moments and prove Theorem 1.2.

**Proof.** We are going to write the null-controllability problem as a problem of moments. From (3.14) we see that  $\tilde{u}(L) = 0$  if, and only if,

$$-\langle \tilde{u}^0, \tilde{S}(L)^* z \rangle_{L^2} = \int_0^L \tilde{U}(t) \overline{\tilde{B}^* \tilde{S}(L-t)^* z} dt, \quad \forall z \in L^2(0, L).$$

Since  $\{\phi_k\}_{k \in \mathbb{Z}}$  is a basis, it is equivalent to

$$-\langle \tilde{u}^0, \tilde{S}(L)^* \phi_k \rangle_{L^2} = \int_0^L \tilde{U}(t) \overline{\tilde{B}^* \tilde{S}(L-t)^* \phi_k} dt, \quad \forall k \in \mathbb{Z}.$$

Since  $\phi_k$  are the eigenfunctions of  $\tilde{A}^*$ , we have  $\tilde{S}(\tau)^* \phi_k = e^{\lambda_k \tau} \phi_k$  and, as a result,

$$-\langle \tilde{u}^0, \phi_k \rangle_{L^2} = \int_0^L e^{-\overline{\lambda_k} t} \tilde{U}(t) \overline{\tilde{B}^* \phi_k} dt, \quad \forall k \in \mathbb{Z}.$$

Since  $\tilde{B}^* \phi_k$  is a nonzero scalar, this is equivalent to

$$c_k = \int_0^L e^{-\overline{\lambda_k} t} \tilde{U}(t) dt, \quad \forall k \in \mathbb{Z}, \tag{3.15}$$

where

$$c_k = -\frac{1}{\overline{\tilde{B}^* \phi_k}} \langle \tilde{u}^0, \phi_k \rangle_{L^2}. \tag{3.16}$$

Now, (3.15)–(3.16) is a standard problem of moments, if the sequence  $\{c_k\}_{k \in \mathbb{Z}}$  belongs to  $\ell^2(\mathbb{Z})$ . Since  $\delta = \inf_{k \in \mathbb{Z}} |\tilde{B}^* \phi_k| > 0$  and  $\{\phi_k\}_{k \in \mathbb{Z}}$  is a Riesz basis (in particular, a Bessel sequence),  $\{c_k\}_{k \in \mathbb{Z}}$  indeed belongs to  $\ell^2(\mathbb{Z})$ :

$$\sum_{k \in \mathbb{Z}} |c_k|^2 \leq \frac{1}{\delta^2} \sum_{k \in \mathbb{Z}} |\langle \tilde{u}^0, \phi_k \rangle_{L^2}|^2 < +\infty.$$

Finally, since  $\{e^{-\overline{\lambda_k} t}\}_{k \in \mathbb{Z}}$  is a Riesz basis (in particular, a Riesz–Fischer sequence), the problem of moments (3.15)–(3.16) has a solution  $\tilde{U} \in L^2(0, L)$  (see Section 3.1).  $\square$

**Remark 8.** Since  $\{e^{-\overline{\lambda_k} t}\}_{k \in \mathbb{Z}}$  is a Riesz basis, the solution  $\tilde{U} \in L^2(0, L)$  to the problem of moments (3.15)–(3.16) is actually unique. This shows that, at least in the case (1.17), the control  $U \in L^2(0, L)$  given by Proposition 2.4 is unique (note the complete analogy with the case  $g = 0$  for which the only null-control possible in the square  $(0, L) \times (0, L)$  is  $U = 0$ ). As a result, there is also only one solution to the kernel equation (2.16) with boundary conditions (2.15) and (2.19).

### Appendix A. Functions of $H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$

This appendix gathers some properties of the functions of  $H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$ . We start with a characterization of the space  $H^1(\mathcal{T}_+)$  (with an obvious analogous statement for

$H^1(\mathcal{T}_-)$ ). We recall that, by definition,  $f \in H^1(\mathcal{T}_+)$  if  $f \in L^2(\mathcal{T}_+)$  and  $f_x, f_y \in L^2(\mathcal{T}_+)$ , where  $f_y \in L^2(\mathcal{T}_+)$  means that there exists  $F \in L^2(\mathcal{T}_+)$  such that

$$\iint_{\mathcal{T}_+} f(x, y)\phi_y(x, y) \, dx dy = - \iint_{\mathcal{T}_+} F(x, y)\phi(x, y) \, dx dy, \quad \forall \phi \in C_c^\infty(\mathcal{T}_+).$$

Such a  $F$  is unique and it is also denoted by  $f_y$ .

**Proposition A.1.** *Let  $f \in L^2(\mathcal{T}_+)$ . The two following properties are equivalent:*

- i)  $f_y \in L^2(\mathcal{T}_+)$ .
- ii) For a.e.  $x \in (0, L)$ , the map

$$f(x) : y \mapsto f(x, y),$$

belongs to  $H^1(x, L)$  and

$$\iint_{\mathcal{T}_+} |f(x)'(y)|^2 \, dy dx < +\infty.$$

Moreover,  $f(x)'(y) = f_y(x, y)$ .

With the help of [Proposition A.1](#) it is not difficult to establish the following.

**Proposition A.2.** *Let  $f \in H^1(\mathcal{T}_-) \cap H^1(\mathcal{T}_+)$  and let us denote by  $f_- \in L^2(\partial\mathcal{T}_-)$  (resp.  $f_+ \in L^2(\partial\mathcal{T}_+)$ ) the trace on  $\mathcal{T}_-$  (resp.  $\mathcal{T}_+$ ) of the restriction of  $f$  to  $\mathcal{T}_-$  (resp.  $\mathcal{T}_+$ ).*

- i) For every  $\varphi \in H^1(0, L)$ , for a.e.  $x \in (0, L)$ , we have

$$\begin{aligned} & \int_0^L f(x, y)\varphi'(y) \, dy \\ &= - \int_0^L f_y(x, y)\varphi(y) \, dy + \left( f_-(x, x) - f_+(x, x) \right)\varphi(x) - f_-(x, 0)\varphi(0) + f_+(x, L)\varphi(L). \end{aligned}$$

- ii) For every  $\varphi \in L^2(0, L)$ , the map

$$\Phi : x \mapsto \int_0^L f(x, y)\varphi(y) \, dy$$

is in  $H^1(0, L)$  with derivative

$$\Phi'(x) = \left( f_-(x, x) - f_+(x, x) \right) \varphi(x) + \int_0^L f_x(x, y) \varphi(y) dy,$$

and traces

$$\Phi(0) = \int_0^L f_+(0, y) \varphi(y) dy, \quad \Phi(L) = \int_0^L f_-(L, y) \varphi(y) dy.$$

**Appendix B. Proof of Proposition 2.1**

This appendix is devoted to the proof of Proposition 2.1.

**Proof.** Let  $u^0 \in L^2(0, L)$  be fixed. Set  $w^0 = P^{-1}u^0 \in L^2(0, L)$  and let  $w \in C^0([0, T]; L^2(0, L))$  be the corresponding solution to (2.2). Let us recall that this means that  $w$  satisfies

$$\begin{aligned} & \int_0^\tau \int_0^L w(t, x) \overline{\left( -\psi_t(t, x) + \psi_x(t, x) \right)} dx dt + \int_0^L w(\tau, x) \overline{\psi(\tau, x)} dx \\ & - \int_0^L w^0(x) \overline{\psi(0, x)} dx = 0, \end{aligned} \tag{B.1}$$

for every  $\psi \in C^1([0, \tau] \times [0, L])$  such that  $\psi(\cdot, 0) = 0$ , and every  $\tau \in [0, T]$ . Note that, by density, it is equivalent to take test functions  $\psi$  in  $L^2(0, \tau; H^1(0, L)) \cap C^1([0, \tau]; L^2(0, L))$ . Let  $u$  be defined by

$$u(t) = Pw(t).$$

Since  $w \in C^0([0, T]; L^2(0, L))$ , it is clear that

$$u \in C^0([0, T]; L^2(0, L)).$$

Moreover, since  $w(T) = 0$ , we also have

$$u(T) = 0.$$

Let us now establish that  $u$  is the solution to (1.13), that is it satisfies

$$\int_0^\tau \int_0^L u(t, x) \left( -\phi_t(t, x) + \phi_x(t, x) - \int_0^L \overline{g(y, x)} \phi(t, y) dy \right) dx dt + \int_0^L u(\tau, x) \overline{\phi(\tau, x)} dx - \int_0^L u^0(x) \overline{\phi(0, x)} dx - \int_0^\tau \Gamma u(t) \overline{\phi(t, L)} dt = 0, \tag{B.2}$$

for every  $\phi \in C^1([0, \tau] \times [0, L])$  such that  $\phi(\cdot, 0) = 0$ , and every  $\tau \in [0, T]$ . Since  $u = Pw$  and  $u^0 = Pw^0$  by definition, we have

$$\int_0^\tau \left\langle u(t), -\frac{d}{dt} \phi(t) - A^* \phi(t) \right\rangle_{L^2} dt + \langle u(\tau), \phi(\tau) \rangle_{L^2} - \langle u^0, \phi(0) \rangle_{L^2} = \int_0^\tau \left\langle w(t), -\frac{d}{dt} P^* \phi(t) - P^* A^* \phi(t) \right\rangle_{L^2} dt + \langle w(\tau), P^* \phi(\tau) \rangle_{L^2} - \langle w^0, P^* \phi(0) \rangle_{L^2}.$$

On the other hand, since  $\phi \in L^2(0, \tau; D(A^*))$ , we can use the hypothesis (2.7) so that

$$-P^* A^* \phi(t) = -A_0^* P^* \phi(t) + P^* \Gamma^* B^* \phi(t).$$

It follows that

$$\int_0^\tau \left\langle u(t), -\frac{d}{dt} \phi(t) - A^* \phi(t) \right\rangle_{L^2} dt + \langle u(\tau), \phi(\tau) \rangle_{L^2} - \langle u^0, \phi(0) \rangle_{L^2} = \int_0^\tau \left\langle w(t), -\frac{d}{dt} P^* \phi(t) - A_0^* P^* \phi(t) \right\rangle_{L^2} dt + \langle w(\tau), P^* \phi(\tau) \rangle_{L^2} - \langle w^0, P^* \phi(0) \rangle_{L^2} + \int_0^\tau \langle w(t), P^* \Gamma^* B^* \phi(t) \rangle_{L^2} dt.$$

Taking the test function  $\psi = P^* \phi$  in (B.1) (note that  $\psi \in L^2(0, \tau; H^1(0, L))$  and satisfies  $\psi(\cdot, 0) = 0$  since  $P^*(D(A^*)) \subset D(A^*)$  by assumption), we see that the second and third lines in the above equality are in fact equal to zero. Taking the adjoints in the remaining term, we obtain (B.2).  $\square$

**Appendix C. Controllability of (3.1) and controllability of (1.1)**

This appendix is devoted to the proof of Proposition 3.1. The proof will use the following two lemmas.

**Lemma C.1.** *Assume that*

$$\tilde{u}^0 \in D(\tilde{A}), \quad \tilde{U} \in H^1(0, T), \quad \tilde{U}(0) = 0. \tag{C.1}$$

*Then, the solution  $\tilde{u}$  to (3.1) belongs to  $H^1((0, T) \times (0, L))$  and satisfies (3.1) almost everywhere.*

**Proof.** It follows from (C.1) and the abstract result [20, Proposition 4.2.10] that

$$\tilde{u} \in C^1([0, T]; L^2(0, L)).$$

On the other hand, by definition, we have

$$\begin{aligned} & \int_0^T \int_0^L \tilde{u}(t, x) \overline{\left( -\phi_t(t, x) + \phi_x(t, x) - \int_0^L \overline{g(y, x)} \phi(t, y) dy \right)} dx dt \\ & - \int_0^L \tilde{u}^0(x) \overline{\phi(0, x)} dx - \int_0^T \tilde{U}(t) \overline{\phi(t, L)} dt = 0, \end{aligned} \tag{C.2}$$

for every  $\phi \in C^1([0, T] \times [0, L])$  such that  $\phi(t, L) = \phi(t, 0)$  and  $\phi(T, x) = 0$ . In particular, for every  $\phi \in C_c^\infty((0, T) \times (0, L))$ , this gives

$$\begin{aligned} & - \int_0^T \int_0^L \tilde{u}(t, x) \overline{\phi_t(t, x)} dx dt + \int_0^T \int_0^L \tilde{u}(t, x) \overline{\phi_x(t, x)} dx dt \\ & - \int_0^T \int_0^L \tilde{u}(t, x) \overline{\left( \int_0^L \overline{g(y, x)} \phi(t, y) dy \right)} dx dt = 0. \end{aligned} \tag{C.3}$$

On the other hand, since  $\tilde{u} \in C^1([0, T]; L^2(0, L))$ , we have

$$\int_0^T \int_0^L \tilde{u}(t, x) \overline{\phi_t(t, x)} dx dt = - \int_0^T \int_0^L \tilde{u}_t(t, x) \overline{\phi(t, x)} dx dt.$$

Coming back to (C.3) we then obtain

$$\int_0^T \int_0^L \tilde{u}(t, x) \overline{\phi_x(t, x)} dx dt = \int_0^T \int_0^L \left( -\tilde{u}_t(t, x) + \int_0^L \tilde{u}(t, y) \overline{g(x, y)} dy \right) \overline{\phi(t, x)} dx dt.$$

Since the map

$$(t, x) \longmapsto -\tilde{u}_t(t, x) + \int_0^L \tilde{u}(t, y)g(x, y) dy$$

belongs to  $L^2((0, T) \times (0, L))$ , this shows that  $\tilde{u}_x \in L^2((0, T) \times (0, L))$  with

$$-\tilde{u}_x(t, x) = -\tilde{u}_t(t, x) + \int_0^L \tilde{u}(t, y)g(x, y) dy, \text{ for a.e. } t \in (0, T), x \in (0, L). \tag{C.4}$$

Now, multiplying (C.4) by  $\bar{\phi} \in C^1([0, T] \times [0, L])$  such that  $\phi(t, L) = \phi(t, 0)$  and  $\phi(T, x) = 0$ , integrating by parts and comparing with (C.2), we obtain

$$\int_0^L \tilde{u}(0, x)\overline{\phi(0, x)} dx + \int_0^T \tilde{u}(t, L)\overline{\phi(t, L)} dt = \int_0^L \tilde{u}^0(x)\overline{\phi(0, x)} dx + \int_0^T \tilde{U}(t)\overline{\phi(t, L)} dt.$$

Taking  $\phi(t, x) = \phi_1(t)\phi_2(x)$  with  $\phi_1 \in C^\infty([0, T])$  such that  $\phi_1(0) = 1$  and  $\phi_1(T) = 0$ , and  $\phi_2 \in C_c^\infty(0, L)$ , we obtain

$$\int_0^L \tilde{u}(0, x)\overline{\phi_2(x)} dx = \int_0^L \tilde{u}^0(x)\overline{\phi_2(x)} dx.$$

Since  $C_c^\infty(0, L)$  is dense in  $L^2(0, L)$ , this gives

$$\tilde{u}(0, x) = \tilde{u}^0(x), \text{ for a.e. } x \in (0, L).$$

Similarly, we can prove that

$$\tilde{u}(t, L) = \tilde{U}(t), \text{ for a.e. } t \in (0, T). \quad \square$$

**Lemma C.2.** Let  $V = D(\tilde{A}) \times \left\{ \tilde{U} \in H^1(0, T) \mid \tilde{U}(0) = 0 \right\}$ . The map

$$\begin{aligned} V &\longrightarrow L^2(0, T) \\ (\tilde{u}^0, \tilde{U}) &\longmapsto \tilde{u}(\cdot, 0), \end{aligned} \tag{C.5}$$

where  $\tilde{u}$  is the solution to (3.1), has a unique continuous extension to  $L^2(0, L) \times L^2(0, T)$ . We shall keep the notation  $\tilde{u}(\cdot, 0)$  to denote this extension.

**Proof.** In virtue of Lemma C.1, for  $(\tilde{u}^0, \tilde{U}) \in V$ , the map (C.5) is well-defined and (3.1) is satisfied almost everywhere. Multiplying (3.1) by  $(L - x)\bar{\tilde{u}}$ , we obtain

$$\begin{aligned} & \int_0^T \frac{1}{2} \frac{d}{dt} \left( \int_0^L (L-x) |\tilde{u}(t,x)|^2 dx \right) dt - \int_0^T \int_0^L (L-x) \frac{1}{2} \partial_x \left( |\tilde{u}(t,x)|^2 \right) dx dt \\ &= \int_0^T \int_0^L \left( \int_0^L g(x,y) \tilde{u}(t,y) dy \right) (L-x) \overline{\tilde{u}(t,x)} dt dx. \end{aligned}$$

Integrating by parts, this gives

$$\begin{aligned} & \frac{1}{2} \int_0^L (L-x) |\tilde{u}(T,x)|^2 dx - \frac{1}{2} \int_0^L (L-x) |\tilde{u}^0(x)|^2 dx \\ &+ \frac{1}{2} L \int_0^T |\tilde{u}(t,0)|^2 dt - \frac{1}{2} \int_0^T \int_0^L |\tilde{u}(t,x)|^2 dx dt \\ &= \int_0^T \int_0^L \left( \int_0^L g(x,y) \tilde{u}(t,y) dy \right) (L-x) \overline{\tilde{u}(t,x)} dt dx. \end{aligned}$$

Using the inequality  $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$  (for  $a, b \geq 0$ ) and the Cauchy–Schwarz’s inequality, we can estimate the term on the right-hand side by  $\|\tilde{u}\|_{C^0([0,T];L^2(0,L))}^2$ . Using then (3.2), we obtain

$$\int_0^T |\tilde{u}(t,0)|^2 dt \leq C \left( \|\tilde{u}^0\|_{L^2(0,L)}^2 + \|\tilde{U}\|_{L^2(0,T)}^2 \right),$$

for some  $C > 0$  (which does not depend on  $\tilde{u}^0$  nor  $\tilde{U}$ ). As a result, the linear map (C.5) is continuous on  $L^2(0, L) \times L^2(0, T)$ . Since  $V$  is dense in  $L^2(0, L) \times L^2(0, T)$ , we can extend this map in a unique continuous way to this space.  $\square$

We can now give the proof of Proposition 3.1:

**Proof.** Let  $\tilde{u}^0 \in L^2(0, L)$  and  $\tilde{U} \in L^2(0, T)$ . Let  $\tilde{u} \in C^0([0, T]; L^2(0, L))$  be the corresponding solution to (3.1). By density of  $D(\tilde{A})$  in  $L^2(0, L)$  and of  $C_c^\infty(0, T)$  in  $L^2(0, T)$ , there exist sequences

$$\tilde{u}_n^0 \in D(\tilde{A}), \quad \tilde{U}_n \in C_c^\infty(0, T),$$

such that

$$\tilde{u}_n^0 \xrightarrow{n \rightarrow +\infty} \tilde{u}^0 \text{ in } L^2(0, L), \quad \tilde{U}_n \xrightarrow{n \rightarrow +\infty} \tilde{U} \text{ in } L^2(0, T). \tag{C.6}$$

Let  $\tilde{u}_n \in C^0([0, T]; L^2(0, L))$  be the solution to



$$\left\{ \begin{aligned} (\tilde{u}_n)_t(t, x) - (\tilde{u}_n)_x(t, x) &= \int_0^L g(x, y)\tilde{u}_n(t, y) dy, & t \in (0, T), x \in (0, L), \\ \tilde{u}_n(t, L) - \tilde{u}_n(t, 0) &= \tilde{U}_n(t), & t \in (0, T), \\ \tilde{u}_n(0, x) &= \tilde{u}_n^0(x), & x \in (0, L). \end{aligned} \right. \tag{C.7}$$

By (3.2) and (C.6), we have

$$\tilde{u}_n \xrightarrow{n \rightarrow +\infty} \tilde{u} \text{ in } C^0([0, T]; L^2(0, L)).$$

On the other hand, by Lemma C.1, we know that

$$\tilde{u}_n \in H^1((0, T) \times (0, L)),$$

and that (C.7) is satisfied almost everywhere. Let  $\tau \in [0, T]$  and  $\phi \in C^1([0, \tau] \times [0, L])$  be such that  $\phi(\cdot, 0) = 0$ . Multiplying (C.7) by  $\overline{\phi}$  and integrating by parts yields

$$\begin{aligned} &\int_0^\tau \int_0^L \tilde{u}_n(t, x) \left( -\phi_t(t, x) + \phi_x(t, x) - \int_0^L \overline{g(y, x)\phi(t, y)} dy \right) dx dt \\ &+ \int_0^L \tilde{u}_n(\tau, x) \overline{\phi(\tau, x)} dx - \int_0^L \tilde{u}_n^0(x) \overline{\phi(0, x)} dx - \int_0^\tau (\tilde{u}_n(t, 0) + \tilde{U}_n(t)) \overline{\phi(t, L)} dt = 0. \end{aligned} \tag{C.8}$$

By Lemma C.2 and (C.6), we know that

$$\tilde{u}_n(\cdot, 0) \xrightarrow{n \rightarrow +\infty} \tilde{u}(\cdot, 0) \text{ in } L^2(0, \tau).$$

Thus, passing to the limit  $n \rightarrow +\infty$  in (C.8), we obtain

$$\begin{aligned} &\int_0^\tau \int_0^L \tilde{u}(t, x) \left( -\phi_t(t, x) + \phi_x(t, x) - \int_0^L \overline{g(y, x)\phi(t, y)} dy \right) dx dt \\ &+ \int_0^L \tilde{u}(\tau, x) \overline{\phi(\tau, x)} dx - \int_0^L \tilde{u}^0(x) \overline{\phi(0, x)} dx - \int_0^\tau (\tilde{u}(t, 0) + \tilde{U}(t)) \overline{\phi(t, L)} dt = 0. \end{aligned}$$

This shows that  $\tilde{u}$  is the (unique) solution of (1.1) with  $u^0 = \tilde{u}^0$  and  $U(t) = \tilde{u}(t, 0) + \tilde{U}(t)$ .  $\square$

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