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Brief paper

Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation ☆

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ABSTRACT

This paper is devoted to a simple and new proof on the optimal finite control time for general linear coupled hyperbolic system by using boundary feedback on one side. The feedback control law is designed by first using a Volterra transformation of the second kind and then using an invertible Fredholm transformation. Both existence and invertibility of the transformations are easily obtained.

1. Introduction

In this paper, we investigate the stabilization of the following $n \times n$ linear coupled hyperbolic system:

$$\begin{cases} u_t(t,x) + \Lambda(x)u_x(t,x) = \Sigma(x)u(t,x), \\ u_-(t,1) = F(u(t)), \ u_+(t,0) = Qu_-(t,0), \\ t \in (0,+\infty), \ x \in (0,1), \end{cases} \quad (1)$$

where $u = (u_-^{\mathrm{T}}, u_+^{\mathrm{T}})^{\mathrm{T}}$ is the state and F is the feedback. We assume that the matrix $\Lambda \in C^1([0,1])^{n\times n}$ is diagonal: $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and such that $\lambda_i(x) \neq 0$ and $\lambda_i(x) \neq \lambda_j(x)$ for every $x \in [0,1]$, for every $i \in \{1,\dots,n\}$ and for every $j \in \{1,\dots,n\} \setminus \{i\}$. Therefore, without loss of generality, we assume that

$$\Lambda = \begin{pmatrix} \Lambda_- & 0 \\ 0 & \Lambda_+ \end{pmatrix},$$

where

$$\Lambda_- = \mathrm{diag}(\lambda_1, \dots, \lambda_m), \quad \Lambda_+ = \mathrm{diag}(\lambda_{m+1}, \dots, \lambda_n),$$

are diagonal submatrices and

$$\lambda_1(x) < \cdots < \lambda_m(x) < 0 < \lambda_{m+1}(x) < \cdots < \lambda_n(x),$$

for all $x \in [0,1]$. Note that we assume that $n \geq 2$ and $m \in \{1,\dots,n-1\}$. Finally, the matrix $\Sigma \in C^0([0,1])^{n\times n}$ couples the equations of the system inside the domain and the constant matrix $Q \in \mathbb{R}^{(n-m)\times m}$ couples the equations of the system on the boundary.

Note that the Riesz representation theorem shows that every bounded linear feedback $F \in \mathcal{L}(L^2(0,1)^n, \mathbb{R}^m)$ has necessarily the form

$$Fu = \left(\sum_{j=1}^n \int_0^1 f_{ij}(x)u_j(x)\,dx \right)_{1\leq i\leq m},$$
$$u = (u_1, \dots, u_n)^{\mathrm{T}} \in L^2(0,1)^n, \quad (2)$$

for some $f_{ij} \in L^2(0,1)$, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$. We can prove that, with this type of boundary conditions, the closed-loop system (1) is well-posed: for every $F \in \mathcal{L}(L^2(0,1)^n, \mathbb{R}^m)$ and $u^0 \in L^2(0,1)^n$, there exists a unique (weak) solution $u \in C^0([0,+\infty); L^2(0,1)^n)$ to

$$\begin{cases} u_t(t,x) + \Lambda(x)u_x(t,x) = \Sigma(x)u(t,x), \\ u_-(t,1) = F(u(t)), \ u_+(t,0) = Qu_-(t,0), \\ u(0,x) = u^0(x), \\ t \in (0,+\infty), \ x \in (0,1). \end{cases} \quad (3)$$

The purpose of this paper is to find a full-state feedback control law F such that the corresponding closed-loop system (1) vanishes after some time, that is such that there exists $T > 0$ such that, for every $u^0 \in L^2(0,1)^n$ for the solution $u \in C^0([0,+\infty); L^2(0,1)^n)$ to

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(3), we have

$$u(t) = 0, \quad \forall t \geq T, \tag{4}$$

and to obtain the best time T such that (4) holds.

The boundary stabilization problem of 1-D hyperbolic systems have been widely investigated in the literature for almost half a century. The pioneer works date back to Rauch and Taylor (1974) and Russell (1978) for linear coupled hyperbolic systems and Greenberg and Li (1984) and Slemrod (1983) for the corresponding nonlinear setting, especially for the quasilinear wave equation. For such systems, many articles are based on the boundary conditions with the following specific form

$$\begin{pmatrix} u_+(t,0) \\ u_-(t,1) \end{pmatrix} = \mathbf{G} \begin{pmatrix} u_+(t,1) \\ u_-(t,0) \end{pmatrix}, \tag{5}$$

where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a suitable smooth vector function. With this boundary condition (5), two methods are distinguished to deal with the stability problem of the linear and nonlinear hyperbolic system. The first one is the so-called characteristic method, which allows us to estimate the related bounds along the characteristic curves. This method was previously investigated in Greenberg and Li (1984) for 2×2 systems and in Li (1994), Qin (1985) and Zhao (1986) for a generalization to $n \times n$ homogeneous nonlinear hyperbolic systems in the framework of C^1 norm. The second one is the control Lyapunov function method, which was introduced in Coron and Bastin (2015), Coron, Bastin, and d'Andréa Novel (2008) and Coron, d'Andréa Novel, and Bastin (2007) to analyze the asymptotic behavior of the nonlinear hyperbolic equations in the context of C^1 and H^2 solutions. Both of these two approaches guarantee the exponential stability of the nonlinear homogeneous hyperbolic systems provided that the boundary conditions are dissipative to some extent. Dissipative boundary conditions are standard static boundary output feedback (that is, a feedback of the state values at the boundaries only). However, there is a drawback of these boundary conditions when inhomogeneous hyperbolic systems are considered, especially the coupling of which are strong enough. In Section 5.6 of the recent monograph (Bastin & Coron 2016), the authors provide a counterexample that shows that there exist linear hyperbolic balance laws, which are controllable by open-loop boundary controls, but are impossible to be stabilized under this kind of boundary feedback.

This limitation can be overcome by using the so-called backstepping method, which connects the original system to a target system with desirable stability properties (e.g. exponential stability) via a Volterra transformation of the second kind. This method was introduced and developed by M. Krstic and his coworkers (see, in particular, the seminal articles (Bošković, Balogh, & Krstić, 2003; Liu, 2003; Smyshlyaev & Krstic, 2004) and the tutorial book (Krstic & Smyshlyaev, 2008)). In Coron, Vazquez, Krstic, and Bastin (2013), the authors designed a full-state feedback control law, with actuation on only one side of the boundary, in order to achieve H^2 exponential stability of the closed-loop 2×2 quasilinear hyperbolic system by using Volterra-type backstepping transformation. Moreover, with this method we can even steer the corresponding linearized hyperbolic system to rest in finite time, that is what is called finite time stabilization. The presented method can also be extended to linear systems with only one negative characteristic velocity (see Di Meglio, Vazquez, & Krstic , 2013). In Hu, Di Meglio, Vazquez, and Krstic (2016), a fully general case of coupled heterodirectional hyperbolic PDEs, allowing an arbitrary number of PDEs convecting in each direction and the boundary controls applied on one side, is presented. The proposed boundary controls also yield the finite-time convergence to zero with the control time given by

$$t_F = \int_0^1 \frac{1}{\lambda_{m+1}(x)}\, dx + \sum_{i=1}^{m} \int_0^1 \frac{1}{|\lambda_i(x)|}\, dx. \tag{6}$$

However, this time t_F is larger than the theoretical optimal one we expect and that is given in Li and Rao (2010), namely

$$T_{\text{opt}} = \int_0^1 \frac{1}{\lambda_{m+1}(x)}\, dx + \int_0^1 \frac{1}{|\lambda_m(x)|}\, dx. \tag{7}$$

In Auriol and Di Meglio (2016), the authors found a minimum time stabilizing controller which makes the coupled hyperbolic system (1) with constant coefficients vanishes after T_{opt} by slightly changing the target system in Hu et al. (2016), in which only local cascade coupling terms are involved in the PDEs.

In this paper, we show that this kind of controller can be established in a much easier way. Inspired by the known results of Hu et al. (2016) and Hu, Vazquez, Di Meglio, and Krstic (2015), we will map the initial coupled hyperbolic system (1) to a new target system in which the cascade coupling terms of the previous works (namely, $G(x)\beta(t,0)$ in Hu et al. (2016) and $\Omega(x)\beta(t,x)$ in Auriol and Di Meglio (2016)) can be completely canceled. Our strategy is to first transform (1) to the target system of Hu et al. (2015) by a Volterra transformation of the second kind, which is always invertible if the kernel belongs to L^2. Then, regarding the target system obtained as the initial hyperbolic system to be studied, by using a Fredholm transformation as introduced in Coron and Lü (2014), we then map this intermediate system to a new target system, vanishing after T_{opt}, without any coupling terms in the PDEs other than a simple trace coupling term. Moreover, the existence and the invertibility of such a transformation will be easily proved (we point out here that these transformations are not always invertible, see Coron, Hu, and Olive (2016), but this will indeed be the case here thanks to the cascade structure of the kernel involved in our Fredholm transformation). Finally, the target system and the original system share the same stability properties due to the invertibility of the transformation.

The main result of this paper is the following:

Theorem 1. *There exists $F \in \mathcal{L}(L^2(0,1)^n, \mathbb{R}^m)$ such that, for every $u^0 \in L^2(0,1)^n$, the solution $u \in C^0([0,+\infty); L^2(0,1)^n)$ to (3) satisfies*

$$u(t) = 0, \quad \forall t \geq T_{\text{opt}},$$

where T_{opt} is given by (7).

Remark 1. We recall that this result has already been obtained in Auriol and Di Meglio (2016), at least in the case of constant matrices Λ and Σ. The approach we shall present below has mainly three advantages compared with the approach of Auriol and Di Meglio (2016). The first one is to map the original system into a target system which is simpler than the one presented in Auriol and Di Meglio (2016) since it has less coupling terms. The second advantage is to provide a method which looks simpler than the method presented in Auriol and Di Meglio (2016), where a successive approximation method is repeatedly used to build the kernels of the transformation, while here we will in a way replace this step by the use of only one single easily computable transformation. The third advantage is in terms of efficiency of the method. To be precise on this particular point, both Auriol and Di Meglio (2016) and our paper are actually based on Theorem 3.3 of Hu et al. (2016), which establishes the existence of a solution to a system of N kernel equations (see equations (36)-(41) in Hu et al. (2016)), where N is the number of equations of the initial system ($n+m$ in Auriol and Di Meglio (2016) and Hu et al. (2016), n in our paper). Moreover, the lines of this system (36)-(41) are in fact uncoupled and they are exactly similar in nature. Therefore, Theorem 3.3 of Hu et al. (2016) applies N times the same existence result with different coefficients. It is precisely this existence result that the authors of Auriol and Di Meglio (2016) and we use (through the result of Hu et al. (2015)). However, in Auriol and Di Meglio (2016), based on an induction argument the authors use one time this result (for the initialization step) and then $N-1$ times the successive approximation method. On the other hand, in our approach, we first use the result of Hu et al. (2015), that is we use N times the same existence result mentioned above and then we use an easily computable transformation. Hence, our method might also be more reliable for practical implementations since we solve N times the same equations and we do it independently while the authors of Auriol and Di Meglio (2016) solve it one time but then they still have to solve $N-1$ equations that are defined recursively, which might lead to larger numerical errors. Finally, it is also worth mentioning that we give a complete proof of the case of space-varying matrices Λ and Σ (based on the result of Hu et al. (2015), but so is Auriol and Di Meglio (2016) as mentioned).

The rest of the paper is organized as follows. In Section 2, we first recall the results of Hu et al. (2015) and then we present a new target system which vanishes after the optimal time T_{opt}. Then, in Section 3, we prove the existence of an invertible Fredholm transformation that maps the target system introduced in Hu et al. (2015) into the new designed target system.

2. New target system

In Hu et al. (2015, Section 2.1) the authors introduced the following target system in the particular case $H=0$:

$$\begin{cases} \gamma_t(t,x)+\Lambda(x)\gamma_x(t,x)=G(x)\gamma(t,0), \\ \gamma_-(t,1)=H(\gamma(t)),\ \ \gamma_+(t,0)=Q\gamma_-(t,0), \\ t\in(0,+\infty),\ x\in(0,1), \end{cases} \tag{8}$$

where $\gamma=(\gamma_-^{\mathrm{T}},\gamma_+^{\mathrm{T}})^{\mathrm{T}}$ is the state and H is a feedback. The matrix $G\in L^\infty(0,1)^{n\times n}$ is a lower triangular matrix with the following structure

$$G=\begin{pmatrix} G_1 & 0 \\ G_2 & 0 \end{pmatrix}, \tag{9}$$

where $G_1\in L^\infty(0,1)^{m\times m}$ has the cascade structure

$$G_1=\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m1} & \cdots & g_{m\,m-1} & 0 \end{pmatrix}, \tag{10}$$

for some $g_{ij}\in L^\infty(0,1)$, $i\in\{2,\dots,m\}$, $j\in\{1,\dots,i-1\}$, and $G_2\in L^\infty(0,1)^{(n-m)\times m}$. We recall that, for every $H\in\mathcal{L}(L^2(0,1)^n,\mathbb{R}^m)$ and $\gamma^0\in L^2(0,1)^n$, there exists a unique (weak) solution $\gamma\in C^0([0,+\infty);L^2(0,1)^n)$ to (8) satisfying $\gamma(0,\cdot)=\gamma^0$.

Taking into account the form of the feedbacks (see (2)) we can use the standard backstepping method and establish the following result, in the exact same way as it was done in Hu et al. (2015) for the case $H=0$:

Lemma 1. *There exist $G\in L^\infty(0,1)^{n\times n}$ with the structure (9)–(10) and an invertible bounded linear map $\mathcal{V}: L^2(0,1)^n\longrightarrow L^2(0,1)^n$ such that, for every $H\in\mathcal{L}(L^2(0,1)^n,\mathbb{R}^m)$, there exists $F\in\mathcal{L}(L^2(0,1)^n,\mathbb{R}^m)$ such that, for every $u^0\in L^2(0,1)^n$, if $\gamma\in C^0([0,+\infty),L^2(0,1)^n)$ denotes the solution to (8) satisfying the initial data $\gamma(0,\cdot)=\mathcal{V}^{-1}u^0$, then*

$$u(t)=\mathcal{V}\gamma(t)$$

is the solution to the Cauchy problem (3).

For the rest of the paper, G is fixed as in Lemma 1.

In Hu et al. (2015), the authors chose the simplest possibility $H=0$ so that, due to the cascade structure (9)–(10), any solution to the resulting system (8) defined at time 0 vanishes after the time t_F given by (6) (see Hu et al., (2015, Proposition 2.1) for more details). However, this appears to be not the best choice since it does not give the expected optimal time T_{opt}. In the present paper, we will show how to properly choose H in order to reduce the vanishing time to T_{opt}. For this purpose, the idea is to apply a second time the backstepping method and find a Fredholm mapping that transforms the previous target system (8) into the following new target system:

$$\begin{cases} z_t(t,x)+\Lambda(x)z_x(t,x)=\widetilde{G}(x)z(t,0), \\ z_-(t,1)=0,\quad z_+(t,0)=Qz_-(t,0), \\ t\in(0,+\infty),\ x\in(0,1), \end{cases} \tag{11}$$

where $z=(z_-^{\mathrm{T}},z_+^{\mathrm{T}})^{\mathrm{T}}$ is the state and $\widetilde{G}\in L^\infty(0,1)^{n\times n}$ is the following matrix

$$\widetilde{G}(x)=\begin{pmatrix} 0 & 0 \\ G_2(x) & 0 \end{pmatrix}, \tag{12}$$

where G_2 is defined in (9). We recall that, for every $z^0\in L^2(0,1)^n$, there exists a unique (weak) solution $z\in C^0([0,+\infty);L^2(0,1)^n)$ to (11) satisfying $z(0,\cdot)=z^0$. Moreover one has the following proposition:

Proposition 1. *For every $z^0\in L^2(0,1)^n$, the solution $z\in C^0([0,+\infty);L^2(0,1)^n)$ to (11) satisfying $z(0,\cdot)=z^0$ verifies $z(t)=0$ for every $t\geq T_{\text{opt}}$.*

Proof. Indeed, using the method of characteristics and the cascade structure (12) of $\widetilde{G}$, one first gets that $z_-(t)=0$ for $t\geq\int_0^1 1/|\lambda_m(x)|\,dx$ and then that $z_+(t)=0$ for $t\geq T_{\text{opt}}$. □

We will prove the following result:

Proposition 2. *There exist an invertible bounded linear map $\mathcal{F}: L^2(0,1)^n\longrightarrow L^2(0,1)^n$ and $H\in\mathcal{L}(L^2(0,1)^n,\mathbb{R}^m)$ such that, for every $\gamma^0\in L^2(0,1)^n$, if $z\in C^0([0,+\infty),L^2(0,1)^n)$ denotes the solution to (11) satisfying the initial data $z(0,\cdot)=\mathcal{F}^{-1}\gamma^0$, then*

$$\gamma(t)=\mathcal{F}z(t),$$

is the solution to (8) satisfying $\gamma(0,\cdot)=\gamma^0$.

Remark 2. In Lemma 1 it is shown that we can reach system (1) from system (8) whatever the feedback H is, provided that F is well chosen. Note that there is no such freedom in Proposition 2 as we need the boundary condition $z_-(t,1)=0$ in a crucial way for the proof, see (18) below.

Combining all the aforementioned results, it is now easy to obtain Theorem 1:

Proof of Theorem 1. Let $\mathcal{F}$ and H be the two mappings provided by Proposition 2 and then let $\mathcal{V}$ and F be the corresponding mappings provided by Lemma 1. Let $z\in C^0([0,+\infty),L^2(0,1)^n)$ be the solution to (11) associated with the initial data $z(0,\cdot)=(\mathcal{V}\circ\mathcal{F})^{-1}u^0$. Then,

$$u(t)=\mathcal{V}\circ\mathcal{F}z(t), \tag{13}$$

is the solution to the Cauchy problem (3). By Proposition 1, we know that $z(t)=0$ for every $t\geq T_{\text{opt}}$ and it readily follows from (13) that $u(t)=0$ for every $t\geq T_{\text{opt}}$ as well. □

Therefore, it only remains to establish Proposition 2. This is achieved in the next section.

3. Existence of an invertible Fredholm transformation

In this section we prove Proposition 2. To this end, we look for a Fredholm transformation $\mathcal{F} : L^2(0,1)^n \longrightarrow L^2(0,1)^n$:

$$\mathcal{F}z(x) = z(x) - \int_0^1 K(x,y)z(y)dy,$$
$$x \in (0,1),\ z \in L^2(0,1)^n, \tag{14}$$

with a kernel $K \in L^2((0,1)\times(0,1))^{n\times n}$ having the following structure:

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{15}$$

in which $K_1 \in L^2((0,1)\times(0,1))^{m\times m}$ is a lower triangular matrix with 0 diagonal entries, that is has the following cascade structure

$$K_1 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ k_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ k_{m1} & \cdots & k_{m\,m-1} & 0 \end{pmatrix}, \tag{16}$$

for some $k_{ij} \in L^2((0,1)\times(0,1))$, $i \in \{2,\dots,m\}$, $j \in \{1,\dots,i-1\}$, yet to be determined. Note that $\mathcal{F}$ is clearly invertible due to this very particular structure (see Appendix for details). Therefore, we only have to check that γ defined by

$$\gamma(t,x) = z(t,x) - \int_0^1 K(x,y)z(t,y)\,dy, \tag{17}$$

is solution to (8) for some $H \in \mathcal{L}(L^2(0,1)^n, \mathbb{R}^m)$ to be determined as well.

Let us first perform some formal computations to derive the equations that the k_{ij} have to satisfy. Taking the derivative with respect to time in (17), using the equation satisfied by z (see the first line of (11)) and integrating by parts yield

$$\begin{aligned}
\gamma_t(t,x) &= z_t(t,x) - \int_0^1 K(x,y)z_t(t,y)\,dy \\
&= -\Lambda(x)z_x(t,x) + \widetilde{G}(x)z(t,0) \\
&\quad + \int_0^1 K(x,y)\Lambda(y)z_y(t,y)\,dy \\
&\quad - \int_0^1 K(x,y)\widetilde{G}(y)z(t,0)\,dy \\
&= -\Lambda(x)z_x(t,x) + \widetilde{G}(x)z(t,0) + K(x,1)\Lambda(1)z(t,1) \\
&\quad - K(x,0)\Lambda(0)z(t,0) - \int_0^1 K_y(x,y)\Lambda(y)z(t,y)\,dy \\
&\quad - \int_0^1 K(x,y)\Lambda_y(y)z(t,y)\,dy \\
&\quad - \int_0^1 K(x,y)\widetilde{G}(y)z(t,0)\,dy.
\end{aligned}$$

Now observe that, since $z_-(t,1) = 0$ and because of the structures of K (see (15)) and $\widetilde{G}$ (see (12)), we have the following two conditions:

$$K(x,1)\Lambda(1)z(t,1) = 0, \tag{18}$$

$$K(x,y)\widetilde{G}(y) = 0.$$

Therefore,

$$\begin{aligned}
\gamma_t(t,x) &= -\Lambda(x)z_x(t,x) + \Big(\widetilde{G}(x) - K(x,0)\Lambda(0)\Big)z(t,0) \\
&\quad - \int_0^1 \Big(K_y(x,y)\Lambda(y) + K(x,y)\Lambda_y(y)\Big)z(t,y)\,dy.
\end{aligned}$$

On the other hand, taking the derivative with respect to space in (17) we have

$$\gamma_x(t,x) = z_x(t,x) - \int_0^1 K_x(x,y)z(t,y)\,dy.$$

As a result, we obtain

$$\begin{aligned}
&\gamma_t(t,x) + \Lambda(x)\gamma_x(t,x) - G(x)\gamma(t,0) \\
&= \Big(\widetilde{G}(x) - K(x,0)\Lambda(0) - G(x)\Big)z(t,0) \\
&\quad - \int_0^1 \Big(K_y(x,y)\Lambda(y) + K(x,y)\Lambda_y(y) \\
&\quad + \Lambda(x)K_x(x,y) - G(x)K(0,y)\Big)z(t,y)\,dy,
\end{aligned}$$

and the right-hand side has to be zero. This yields to the following kernel system

$$\begin{aligned}
&\Lambda(x)K_x(x,y) + K_y(x,y)\Lambda(y) \\
&\quad + K(x,y)\Lambda_y(y) - G(x)K(0,y) = 0
\end{aligned}$$

with the condition

$$K(x,0) = (\widetilde{G}(x) - G(x))\Lambda^{-1}(0).$$

In order to guarantee the well-posedness of the system satisfied by K, we impose the following extra condition:

$$K_-(0,y) = 0, \tag{19}$$

(where K_- denotes the submatrix containing the first m rows of K), which turns out to also imply the following, because of the structures of G (see (9)) and K (see (15)),

$$G(x)K(0,y) = 0,$$

and therefore makes the kernel system much simpler to solve. To summarize, K will satisfy the system

$$\begin{cases}
\Lambda(x)K_x(x,y) + K_y(x,y)\Lambda(y) + K(x,y)\Lambda_y(y) = 0, \\
K_-(0,y) = 0, \\
K(x,0) = (\widetilde{G}(x) - G(x))\Lambda^{-1}(0), \\
x,y \in (0,1).
\end{cases}$$

Note that the structure (15) of K, (17) and (19) imply that

$$\gamma(t,0) = z(t,0).$$

Therefore, the boundary condition at $x = 0$ for γ is automatically guaranteed:

$$\gamma_+(t,0) = z_+(t,0) = Qz_-(t,0) = Q\gamma_-(t,0).$$

Now, because of the structures of K, $\widetilde{G}$ and G given in (15), (12) and (9) respectively, the system for K translates into the following system for K_1:

$$\begin{cases}
\Lambda_-(x)(K_1)_x(x,y) + (K_1)_y(x,y)\Lambda_-(y) \\
\qquad\qquad + K_1(x,y)(\Lambda_-)_y(y) = 0, \\
K_1(0,y) = 0, \\
K_1(x,0) = -G_1(x)\Lambda_-^{-1}(0), \\
x,y \in (0,1).
\end{cases}$$

Regarding y as the time parameter, this is a standard time-dependent uncoupled hyperbolic system with only positive speeds $\lambda_i(x)/\lambda_j(y) > 0$, $i,j \in \{1,\dots,m\}$, and therefore it admits a unique (weak) solution $K_1 \in L^2((0,1)\times(0,1))^{m\times m}$. Actually, using the method of characteristics, we see that the solution is explicitly given by

$$k_{ij}(x,y) = \frac{g_{ij}\big(\phi_i^{-1}(\phi_i(x) - \phi_j(y))\big)}{-\lambda_j(y)}, \tag{20}$$

if $i \in \{2, \dots, m\}, j \in \{1, \dots, i-1\}$ and $\phi_i(x) \le \phi_j(y)$, and $k_{ij}(x, y) = 0$ otherwise, where

$$\phi_i(x) = \int_0^x \frac{1}{\lambda_i(\xi)} d\xi, \quad i \in \{1, \dots, m\}.$$

Note that ϕ_i is indeed invertible since it is a strictly monotonically decreasing continuous function of x. Finally, we readily see from (20) that

$$K_1(1, \cdot) \in L^2(0, 1)^{m\times m},$$

so that the map $H : L^2(0, 1)^n \longrightarrow \mathbb{R}^m$ given by

$$H\gamma = -\int_0^1 K_1(1, y)\big[\mathcal{F}^{-1}\gamma\big]_-(y)\, dy, \quad \gamma \in L^2(0, 1)^n, \tag{21}$$

is well-defined and $H \in \mathcal{L}(L^2(0, 1)^n, \mathbb{R}^m)$. This concludes the proof of Proposition 2. □

Remark 3. Note that the feedback H we built is explicit once the function G_1 that appears in system (8) is given. Indeed, first we see from (20) that $K_1(1, \cdot)$ is uniquely determined by the function G_1. On the other hand, the inverse Fredholm transformation $\mathcal{F}^{-1}$ can easily be computed in terms of k_{ij} (and thus G_1), see the proof of Lemma 2 in the appendix below.

4. Conclusion

In this paper, we provide an alternative way to Auriol and Di Meglio (2016) to realize the stabilization of 1-D linear hyperbolic balance laws with boundary feedback on one side of the system. The zero-equilibrium of the system is reached in the best possible time T_{opt}. The feedback control is designed by first using a Volterra transformation of the second kind as in Hu et al. (2015) and then an invertible Fredholm transformation, in which explicit formulas of the kernels can be obtained. Our main improvement is to consider a target system which is simpler than the one used in Auriol and Di Meglio (2016) and to also provide a method that might be more efficient than the one presented in Auriol and Di Meglio (2016) (which is also based on the results used in Hu et al. (2015)). A comparison between the proposed results and Auriol and Di Meglio (2016) in term of control efforts, size of the kernels, transient-performance would be of interest. Let us conclude this paper by mentioning that an ultimate goal would be to give a complete description of the target systems that can be achieved by general invertible linear transformations. In connection with this goal, let us mention that it is proved in Coron (2015) that, for every linear finite dimensional control system $\dot{y} = Ay + Bu$, the controllability (which is a necessary and sufficient condition for the rapid stabilization) implies that, for every $\lambda \in \mathbb{R}$, the target system $\dot{y} = Ay - \lambda y + Bu$ can always be achieved by a linear transformation on the state and on the control.

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Appendix. Invertibility of the Fredholm transformation

For the completeness we prove in this appendix the invertibility of the Fredholm transformation $\mathcal{F}$.

Lemma 2. *For any given $K \in L^2((0, 1) \times (0, 1))^{n\times n}$ with the cascade structure (15)–(16), the transformation $\mathcal{F}$ defined by (14) is invertible. Moreover, its inverse has the same form:*

$$\mathcal{F}^{-1}\gamma(x) = \gamma(x) - \int_0^1 \widetilde{\Theta}(x, y)\gamma(y)dy,$$
$$x \in (0, 1),\ \gamma \in L^2(0, 1)^n,$$

for some $\widetilde{\Theta} \in L^2((0, 1) \times (0, 1))^{n\times n}$ with the same structure as K, that is,

$$\widetilde{\Theta} = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix},$$

in which $\Theta \in L^2((0, 1)\times(0, 1))^{m\times m}$ is a lower triangular matrix with 0 diagonal entries as K_1:

$$\Theta = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \theta_{2\,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \theta_{m\,1} & \cdots & \theta_{m\,m-1} & 0 \end{pmatrix},$$

for some $\theta_{ij} \in L^2((0, 1) \times (0, 1))$, $i \in \{2, \dots, m\}$ and $j \in \{1, \dots, i-1\}$.

Proof of Lemma 2. Let $\gamma = \mathcal{F}z$, where $z \in L^2(0, 1)^n$ is given. Thanks to (15) and (17), we have

$$z_i = \gamma_i, \quad \forall i \in \{m+1, \dots, n\}.$$

On the other hand, thanks to (16) and (17), we have

$$\begin{cases} \gamma_1 = z_1, \\ \gamma_i = z_i - \displaystyle\sum_{j=1}^{i-1} \int_0^1 k_{ij}(\cdot, y)z_j(y)\, dy, \ \forall i \in \{2, \dots, m\}. \end{cases}$$

By induction we readily see that

$$\begin{cases} z_1 = \gamma_1, \\ z_i = \gamma_i - \displaystyle\sum_{j=1}^{i-1} \int_0^1 \theta_{ij}(\cdot, y)\gamma_j(y)\, dy, \ \forall i \in \{2, \dots, m\}, \end{cases}$$

for some $\theta_{ij} \in L^2(0, 1)$ depending only on k_{pq} for $p \in \{1, \dots, i\}$ and $q \in \{1, \dots, j\}$. This proves Lemma 2. □

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