Controllability of parabolic systems

Guillaume OLIVE

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Outline

1. Review of the controllability for the heat equation

2. Controllability of a parabolic system
   - Carleman estimates
   - Spectral conditions
   - Geometric control conditions
   - Minimal time of control
   - Boundary controllability

3. Comments

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Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ a smooth nonempty bounded open connected subset and set

$$Q_T = (0, T) \times \Omega, \quad \Sigma_T = (0, T) \times \partial\Omega.$$

We consider the heat equation:

$$\begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{1}$$

where

- y is the state y^0 the initial data,
- $v \in L^2(Q_T)$ is the control,
- the nonempty open subset $\omega \subset \Omega$ localises in space the control.

Well-posedness: for every $y^0 \in L^2(\Omega)$ and $v \in L^2(Q_T)$, there exists a unique

$$y \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)),$$

(weak) solution to (1). Moreover, this solution continuously depends on (y^0, v).

Controllability

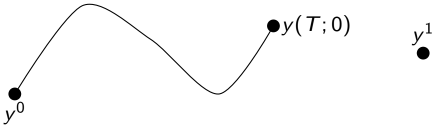

Figure: Uncontrolled trajectory

y^0: initial data, y^1: target, $y(T;v)$: solution to (1) at time T with the control v.

Definition (Notions of controllability)

- (1) is null-controllable if: $\forall y^0 \in L^2(\Omega)$, $\exists v \in L^2(Q_T)$ such that $y(T) = 0$.
- (1) is approximately controllable if: $\forall \varepsilon > 0$, $\forall y^0, y^1 \in L^2(\Omega)$, $\exists v \in L^2(Q_T)$ such that $\left\| y(T) - y^1 \right\|_{L^2} \leq \varepsilon$.

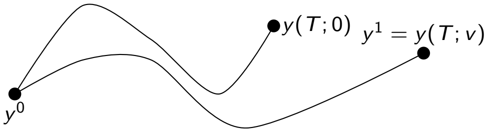

Figure: Trajectory controlled exactly

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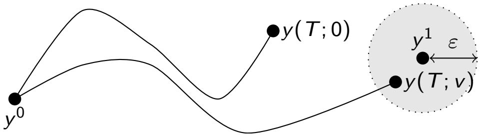

Figure: Trajectory controlled approximately

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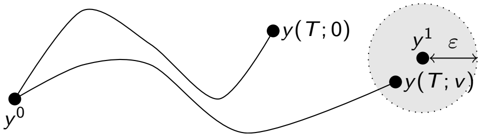

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For the heat equation (1),

- exact controllability to a state $y^1 \notin C^\infty$ is impossible (regularizing effect in $\Omega \backslash \overline{\omega}$),
- null-controllability implies approximate controllability (backward uniqueness of the adjoint system),
- approximate controllability does not depend on the time of control T (analyticity in time of the adjoint system).

Duality controllability - observability

Let us introduce the adjoint system to (1), that is the backward (in time) equation

$$\begin{cases} -\partial_t z - \Delta z = 0 & \text{in } Q_T, \\ z = 0 & \text{sur } \Sigma_T, \\ z(T) = z^0 & \text{in } \Omega. \end{cases} \tag{2}$$

Multiplying (1) by z and performing integrations by parts, we have the fundamental relation

$$\left\langle y(T), z^0 \right\rangle_{L^2} - \left\langle y^0, z(0) \right\rangle_{L^2} = \int_0^T \langle v(t), 1_\omega z(t) \rangle_{L^2} \, dt, \quad \forall y^0, z^0 \in L^2(\Omega),$$

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from which we deduce

Theorem (S. Dolecki and D.L. Russell (1977))

1. (1) *is null-controllable at time* T *if, and only if,* (2) *satisfies*

$$\exists C_T > 0, \forall z^0 \in L^2(\Omega), \quad \|z(0)\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|1_\omega z(t)\|^2_{L^2(\Omega)} \, dt. \tag{Obs}$$

2. (1) *is approximately controllable if, and only if,* (2) *satisfies*

$$\forall z^0 \in L^2(\Omega), \quad \Big(1_\omega z(t) = 0, \quad a.e. \ t \in (0, T)\Big) \Longrightarrow z^0 = 0. \tag{UCP}$$

(Obs) is called observability inequality. (UCP) is called unique continuation property.

A general result of H.O. FATTORINI

Actually, we have a simpler characterization of (UCP):

Theorem (H.O. FATTORINI (1966), particular case)

The (UCP) *property*

$$\forall z^0 \in L^2(\Omega), \quad \left(1_\omega z(t) = 0, \quad a.e. \ t \in (0,T)\right) \Longrightarrow z^0 = 0,$$

has to be check only in the eigenspaces of the operator Δ, that is, for

$$z^0 \in \ker(-\lambda_k - \Delta), \quad \forall k \in \mathbb{N}^*,$$

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where $-\lambda_k$ *are the Dirichlet eigenvalues of* Δ*.*

- More formally, this reads

$$\ker(-\lambda_k - \Delta) \cap \ker 1_\omega = \{0\}, \quad \forall k \in \mathbb{N}^*.$$

In "finite dimension" this theorem is also known as the Hautus test (1969).

- In fact, this theorem holds in a very large setting (including most of parabolic systems).

Theorem

For any $T > 0$ and any control domain ω, (1) is null-controllable at time T.

This can be proved using

1. the method of moments (proof in dimension 1),
   H.O. Fattorini and D.L. Russell (1971),
2. elliptic Carleman estimates,
   G. Lebeau and L. Robbiano (1995),
3. parabolic Carleman estimates,
   A. Fursikov and O. Yu. Imanuvilov (1996),
4. the transmutation method (from the wave equation to the heat equation),
   S. Everdoza and E. Zuazua (2011) (introduced by L. Miller (2006)).

Let $\theta(t) = \frac{1}{t(T-t)}$.
Let $\omega \subset\subset \Omega$. There exists a positive function $\gamma \in C^2(\overline{\Omega})$ such that

Theorem (A. Fursikov and O. Yu. Imanuvilov (1996))

There exist $C > 0$ and $s_0 \geq 1$ such that

$$I(s;z) \stackrel{\text{def}}{=} \iint_{Q_T} \frac{1}{s\theta} e^{-2s\theta\gamma} |\partial_t z|^2 \, dx\, dt + \iint_{Q_T} \frac{1}{s\theta} e^{-2s\theta\gamma} |\Delta z|^2 \, dx\, dt + \iint_{Q_T} s\theta e^{-2s\theta\gamma} |\nabla z|^2 \, dx\, dt + \iint_{Q_T} (s\theta)^3 e^{-2s\theta\gamma} |z|^2 \, dx\, dt \leq C \left(\iint_{(0,T)\times\omega} (s\theta)^3 e^{-2s\theta\gamma} |z|^2 \, dx\, dt + \iint_{Q_T} e^{-2s\theta\gamma} |-\partial_t z - \Delta z|^2 \, dx\, dt \right),$$

for every $s \geq s_0$ and $z \in C^\infty(\overline{Q_T})$ such that $z = 0$ on Σ_T.

Applying the Carleman estimate to the solution z of the adjoint system (2) gives

$$\iint_{Q_T} (s\theta)^3 e^{-2s\theta\gamma} |z|^2 \, dx \, dt \leq C \iint_{(0,T)\times\omega} (s\theta)^3 e^{-2s\theta\gamma} |z|^2 \, dx \, dt.$$

Since

$$\begin{cases} (s\theta)^3 e^{-2s\theta\gamma} \in L^\infty(Q_T), \\ (s\theta)^3 e^{-2s\theta\gamma} \geq C \text{ in } \left(\frac{T}{4}, \frac{3T}{4}\right) \times \Omega, \end{cases}$$

we obtain

$$\iint_{\left(\frac{T}{4},\frac{3T}{4}\right)\times\Omega} |z|^2 \, dx \, dt \leq C \iint_{(0,T)\times\omega} |z|^2 \, dx \, dt.$$

On the other hand, the growth of the energy $t \mapsto \|z(t)\|^2_{L^2(\Omega)}$ gives

$$\frac{T}{2} \|z(0)\|^2_{L^2(\Omega)} \leq \iint_{(T/4,3T/4)\times\Omega} |z|^2 \, dx \, dt.$$

□

Boundary controllability

We can also be interested in the boundary controllability of the heat equation:

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } Q_T, \\ y = 1_\gamma v & \text{on } \Sigma_T, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{3}$$

where γ is a nonempty relative open subset of $\partial\Omega$.
But in fact, the null-controllability of (3) is a consequence of the one of (1) (and vice versa).

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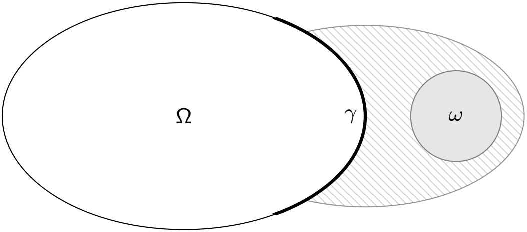
where γ is a nonempty relative open subset of $\partial\Omega$.
But in fact, the null-controllability of (3) is a consequence of the one of (1) (and vice versa).
Proof:

1. extend the domain,
2. take the trace of the controlled extended solution as control.



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To summarize:

- The heat equation is null-controllable at any time T.
- The heat equation is null-controllable whatever the control domain ω is.
- For the heat equation, the boundary controllability is a consequence of the distributed controllability.
- Methods to prove the null-controllability: the method of moments, Carleman estimates, method of transmutation.
- Methods to prove the approximate controllability: theorem of Fattorini.

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Toy model of parabolic systems

We will focus on the distributed controllability of the following 2×2 system by 1 control:

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 1_\omega v & \text{in } Q_T, \\ \partial_t y_2 - \Delta y_2 = a_{21}(x) y_1 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } \Omega, \end{cases} \tag{Syst}$$

where

- (y_1, y_2) is the state and $(y_1^0, y_2^0) \in L^2(\Omega)^2$ the initial data,
- $v \in L^2(Q_T)$ is the control,
- $\omega \subset \Omega$ localises in space the control,
- $a_{21} \in L^\infty(\Omega)$ couples the second equation to the first one.

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Remark

The controllability of (Syst) *by 2 controls (one on each equation) is easy (apply Carleman estimates to both equations and add them up).*

CARLEMAN ESTIMATES

Control domain intersecting the support of the coupling

Theorem (L. de Teresa (2000))

Let $T > 0$. Assume that there exist a nonempty open subset $\omega' \subset\subset \omega$ and $c > 0$ such that

$$a_{21}(x) \geq c, \quad \text{a.e. in } \omega'.$$

Then, (Syst) *is null-controllable at time T.*

This hypothesis thus requires that $\omega \cap \operatorname{supp} a_{21} \neq \emptyset$.

The adjoint system of (Syst) is

$$\begin{cases} -\partial_t z_1 - \Delta z_1 = a_{21}(x) z_2 & \text{in } Q_T, \\ -\partial_t z_2 - \Delta z_2 = 0 & \text{in } Q_T, \\ z_1 = z_2 = 0 & \text{on } \Sigma_T, \end{cases} \tag{4}$$

and the corresponding observability inequality reads

$$\|z_1(0)\|^2_{L^2(\Omega)} + \|z_2(0)\|^2_{L^2(\Omega)} \leq C \int_0^T \|1_\omega z_1(t)\|^2_{L^2(\Omega)}\, dt. \tag{5}$$

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Let $\omega'' \subset\subset \omega'$. We apply the Carleman estimates in ω'' to both equations of (4):

$$I(s; z_1) + I(s; z_2) \leq C \Bigg(\iint_{(0,T)\times\omega''} (s\theta)^3 e^{-2s\theta\gamma} |z_1|^2 \, dx \, dt + \iint_{(0,T)\times\omega''} (s\theta)^3 e^{-2s\theta\gamma} |z_2|^2 \, dx \, dt \Bigg).$$

We have to remove the second term.

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We have to remove the second term.
Let us prove that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\iint_{(0,T)\times\omega''} (s\theta)^3 e^{-2s\theta\gamma} |z_2|^2 \, dx\, dt \leq \varepsilon I(s, z_2) + C_\varepsilon \iint_{(0,T)\times\omega'} (s\theta)^7 e^{-2s\theta\gamma} |z_1|^2 \, dx\, dt.$$

To obtain the observability inequality from these estimates is then classical.

Proof 2/3

Let $\xi \in C^\infty(\mathbb{R})$ be a cut-off function (to avoid the boundary terms in space):

$$\xi = 0 \text{ in } \mathbb{R}\backslash\omega', \quad \xi = 1 \text{ in } \omega'', \quad 0 \leq \xi \leq 1 \text{ in } \mathbb{R}.$$

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$$\iint_{(0,T)\times\omega''} (s\theta)^3 e^{-2s\theta\gamma} |z_2|^2 \, dx\, dt \leq \frac{1}{c} \iint_{(0,T)\times\omega'} (s\theta)^3 e^{-2s\theta\gamma} \xi a_{21} |z_2|^2 \, dx\, dt.$$

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Multiplying the equation $-\partial_t z_1 - \Delta z_1 = a_{21}(x) z_2$ by $(s\theta)^3 e^{-2s\theta\gamma} \xi z_2$ leads to

$$\iint_{(0,T)\times\omega'} (s\theta)^3 e^{-2s\theta\gamma} \xi a_{21} |z_2|^2 \, dx \, dt = \iint_{(0,T)\times\omega'} (s\theta)^3 e^{-2s\theta\gamma} \xi (-\partial_t z_1 z_2 - \Delta z_1 z_2) \, dx \, dt.$$

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Let us estimate $-\iint_{(0,T)\times\omega'} (s\theta)^3 e^{-2s\theta\gamma} \xi \partial_t z_1 z_2 \, dx\, dt$. An integration by parts gives

$$\begin{aligned} -\iint_{(0,T)\times\omega'} (s\theta)^3 e^{-2s\theta\gamma} \xi \partial_t z_1 z_2 \, dx\, dt \quad &= \quad \iint_{(0,T)\times\omega'} \partial_t \left((s\theta)^3 e^{-2s\theta\gamma} \xi \right) z_2 z_1 \, dx\, dt \\ &+ \iint_{(0,T)\times\omega'} (s\theta)^3 e^{-2s\theta\gamma} \xi (\partial_t z_2) z_1 \, dx\, dt. \end{aligned}$$

Let us estimate $\iint_{(0,T)\times\omega'}(s\theta)^3 e^{-2s\theta\gamma}\xi(\partial_t z_2)z_1\,dx\,dt$.
Using Young's inequality

$$(s\theta)^3\,|\partial_t z_2|\,|z_1| \leq \varepsilon\frac{1}{s\theta}\,|\partial_t z_2|^2 + \frac{1}{\varepsilon}(s\theta)^7\,|z_1|^2\,,$$

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we obtain

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The other terms can be handled the same way. □

Remark

From now on, we don't assume anymore that $\omega \cap \operatorname{supp} a_{21} \neq \emptyset$.

SPECTRAL CONDITIONS

Spectral properties

Let us denote

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ a_{21} & \Delta \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = (H^2 \cap H_0^1)^2.$$

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$$\mathcal{A}^* = \begin{pmatrix} \Delta & a_{21} \\ 0 & \Delta \end{pmatrix}, \quad \mathcal{D}\left(\mathcal{A}^*\right) = (H^2 \cap H_0^1)^2.$$

- The spectrum of $\mathcal{A}^*$ is

$$\sigma\left(\mathcal{A}^*\right) = \left\{-\lambda_k\right\}_{k \in \mathbb{N}^*}.$$

Let us denote

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ a_{21} & \Delta \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = (H^2 \cap H_0^1)^2.$$

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- The spectrum of $\mathcal{A}^*$ is

$$\sigma(\mathcal{A}^*) = \{-\lambda_k\}_{k \in \mathbb{N}^*}.$$

- The eigenspaces of $\mathcal{A}^*$ can be decomposed as the orthogonal direct sum

$$\ker(-\lambda_k - \mathcal{A}^*) = U_k \oplus^{\perp} W_k,$$

where

$$U_k = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \middle| \, u \in \ker(-\lambda_k - \Delta) \right\}, \quad W_k = \left\{ \begin{pmatrix} S_k(a_{21} w) \\ w \end{pmatrix} \middle| \, w \in \ker(-\lambda_k - \Delta) \cap \ker(\Pi_k a_{21}) \right\},$$

where $S_k : f \in \ker \Pi_k \longmapsto u \in \ker \Pi_k$ with u the unique solution (in $\ker \Pi_k$) of

$$\begin{cases} (-\lambda_k - \Delta)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Sufficient condition of approximate controllability

Theorem (O. Kavian and L. de Teresa (2010); G. O. (2014))

Assume that

$$\ker(-\lambda_k - \Delta) \cap \ker(\Pi_k a_{21}) = \{0\}, \quad \forall k \in \mathbb{N}^*. \tag{6}$$

Then, (Syst) *is approximately controllable.*

- In general, (6) is not a necessary condition, see after.

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- (6) can be reformulated into

$$\det\left(\int_\Omega a_{21}\phi_{k,i}\phi_{k,j}\,dx\right)_{1\leq i,j\leq m_k} \neq 0, \quad \forall k \in \mathbb{N}^*, \tag{7}$$

where $\phi_{k,1},\dots,\phi_{k,m_k}$ is a basis of $\ker(\lambda_k - \Delta)$.

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where $\phi_{k,1}, \ldots, \phi_{k,m_k}$ is a basis of $\ker(\lambda_k - \Delta)$.

- In the one-dimensional case $\Omega = (0,1)$ (denoting $\phi_{k,1} = \phi_k$ since $m_k = 1$)

$$\mathcal{I}_k = \int_0^1 a_{21} (\phi_k)^2 \, dx \neq 0, \quad \forall k \in \mathbb{N}^*.$$

Proof of G. O.

Let

$$\mathcal{B} = \begin{pmatrix} 1_\omega \\ 0 \end{pmatrix}, \quad \mathcal{D}\left(\mathcal{B}\right) = L^2(\Omega)^2.$$

By the theorem of Fattorini, the approximate controllability is equivalent to

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As a result

$$\begin{pmatrix} u \\ w \end{pmatrix} \in \ker(-\lambda_k - \mathcal{A}^*) \cap \ker \mathcal{B}^* \Longleftrightarrow (w \equiv 0 \quad \text{and} \quad u \in \ker(-\lambda_k - \Delta) \cap 1_\omega).$$

The unique continuation for a single equation then gives

$$u \equiv 0.$$

□

GEOMETRIC CONTROL CONDITIONS

Reduction to a nonhomogeneous scalar elliptic problem

- In this part, we focus again on the approximate controllability.
- By the theorem of Fattorini, we have to study the property

$$\left.\begin{array}{rl} -\Delta u - \lambda_k u = a_{21} w & \text{in } \Omega \\ -\Delta w - \lambda_k w = 0 & \text{in } \Omega \\ u = 0 & \text{in } \omega \end{array}\right\} \Longrightarrow u \equiv w \equiv 0.$$

We treat this problem as a nonhomogeneous scalar equation:

$$-\Delta u - \lambda_k u = F \quad \text{in } \Omega,$$

where $F = a_{21} w$ is known.

Notations

- From now on, $\Omega = (0, 1)$.
- $\omega \subset \Omega$ is still the control domain and ω is not necessarily connected.
- ϕ_k denotes again the eigenfunctions of ∂_x^2 associated with $-\lambda_k$.

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- Let $\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ be the set of connected component of $\overline{\Omega\backslash\omega}$.
- For every $C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right)$ and $F \in L^2(\Omega)$, let $M_k(F,C)$ be the vector of $\mathbb{R}^2$ defined by

$$M_k(F,C) = \begin{pmatrix} \int_C F\phi_k \, dx \\ 0 \end{pmatrix} \text{ if } C \cap \partial\Omega \neq \emptyset, \quad M_k(F,C) = \begin{pmatrix} \int_C F\phi_k \, dx \\ \int_C F\phi_k' \, dx \end{pmatrix} \text{ si } C \cap \partial\Omega = \emptyset,$$

For instance,

$$\Longrightarrow M_k(F,C) = \begin{pmatrix} \int_C F\phi_k \, dx \\ 0 \end{pmatrix}, \quad \forall C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right).$$

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For instance,

$$\Longrightarrow M_k(F,C) = \begin{pmatrix} \int_C F\phi_k \, dx \\ 0 \end{pmatrix}, \quad \forall C \in \mathcal{C}\left(\overline{\Omega\backslash\omega}\right).$$

- Finally, for every $F \in L^2(\Omega)$ we define the following family of vectors of $\mathbb{R}^2$:

$$\mathcal{M}_k(F,\omega) = (M_k(F,C))_{C\in\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)} \in (\mathbb{R}^2)^{\mathcal{C}\left(\overline{\Omega\backslash\omega}\right)}.$$

Theorem (F. Boyer and G.O. (2014))

Let $k \in \mathbb{N}^*$ *and* $F \in L^2(\Omega)$*. We have*

$$\exists u \in H^2 \cap H_0^1, \quad \begin{cases} -\partial_x^2 u - k^2\pi^2 u = F & \text{in } \Omega, \\ u = 0 & \text{in } \omega, \end{cases}$$

if, and only if,

$$\begin{cases} F = 0 & \text{in } \omega, \\ \mathcal{M}_k(F,\omega) = 0. \end{cases}$$

Theorem (F. Boyer and G. O. (2014))

Assume that $\omega \cap \operatorname{supp} a_{21} = \emptyset$. *Then,* (Syst) *is approximately controllable if, and only if,*

$$\mathcal{M}_k\left(a_{21}\phi_k, \omega\right) \neq 0, \quad \forall k \in \mathbb{N}^*.$$

Simple conditions for the approximate controllability

Corollary (F. Boyer and G. O. (2014))

Assume that $\omega \cap \operatorname{supp} a_{21} = \emptyset$.

1. ***Sufficient condition:*** (Syst) *is approximately controllable if* a_{21} *satisfies*

$$\mathcal{I}_k = \int_0^1 a_{21}(\phi_k)^2 \, dx \neq 0, \quad \forall k \in \mathbb{N}^*. \tag{8}$$

2. ***Necessary condition:*** *if* (Syst) *is approximately controllable and* $\omega, \operatorname{supp} a_{21}$ *are connected, then* (8) *has to hold.*

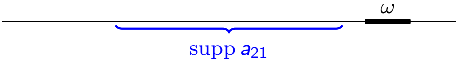
In general, (8) is not necessary, see right after.

Role of the geometry of the control domain

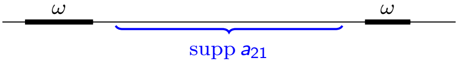
Let us take a look at the particular case

$$a_{21}(x) = \left(x - \frac{1}{2}\right) 1_{\mathcal{O}}(x), \quad \mathcal{O} = \operatorname{supp} a_{21} = \left(\frac{1}{4}, \frac{3}{4}\right).$$

Consider the two following geometric configurations for ω:



(a) ω is connected



(b) ω is not connected

- (Syst) **is not** approximately controllable in configuration (a).
- (Syst) is approximately controllable in configuration (b).

MINIMAL TIME OF CONTROL

Theorem (F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa (2014))

Let $\omega = (\alpha, \beta) \subset \Omega = (0,1)$*. Assume that*

$$\mathcal{I}_k = \int_0^1 a_{21}(\phi_k)^2\,dx \neq 0, \quad \forall k \in \mathbb{N}^*. \tag{9}$$

Let

$$T_0 = \limsup_{k\to+\infty} \frac{\log \frac{1}{|\mathcal{I}_k|}}{k^2\pi^2} \quad (\textit{possibly } T_0 = +\infty).$$

1. *If* $T > T_0$*, then* (Syst) *is null-controllable at time* T*.*
2. *If* $T < T_0$*, and*

$$\operatorname{supp} a_{21} \subset (0,\alpha) \quad \textit{or} \quad \operatorname{supp} a_{21} \subset (\beta,1),$$

   then (Syst) *is not null-controllable at time* T*.*

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1. *If* $T > T_0$, *then* (Syst) *is null-controllable at time* T.
2. *If* $T < T_0$, *and*

$$\operatorname{supp} a_{21} \subset (0, \alpha) \quad \textit{or} \quad \operatorname{supp} a_{21} \subset (\beta, 1),$$

   then (Syst) *is not null-controllable at time* T.

- By Riemann-Lebesgue's lemma, $(\mathcal{I}_k)_k$ converges to $\int_0^1 a_{21} \, dx$. Thus, $\int_0^1 a_{21} \, dx \neq 0 \Longrightarrow T_0 = 0$.
- We will give a proof of the second point.
- The first point can be proved using the method of moments, looking for controls in the form $v(t, x) = v_1(t) v_2(x)$ with v_2 supported in ω.

A comparison between null and approximate controllability:

Remark

(Syst) *can be approximately controllable but never null-controllable.*

The (approximate controllability) condition (9) implies that

$$\ker(-k^2\pi^2 - \mathcal{A}^*)^2 = \mathrm{span}\left\{ \begin{pmatrix} S_k(a_{21}\phi_k - \mathcal{I}_k\phi_k) \\ \phi_k \end{pmatrix} \right\} \oplus^\perp \ker(-k^2\pi^2 - \mathcal{A}^*),$$

and

$$\ker(-k^2\pi^2 - \mathcal{A}^*) = \mathrm{span}\left\{ \begin{pmatrix} \phi_k \\ 0 \end{pmatrix} \right\}.$$

Generalized eigenspaces and key Lemma

The (approximate controllability) condition (9) implies that

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and

$$\ker(-k^2\pi^2 - \mathcal{A}^*) = \mathrm{span}\left\{\begin{pmatrix} \phi_k \\ 0 \end{pmatrix}\right\}.$$

Lemma

Assume that $\mathrm{supp}\, a_{21} \subset (0,\alpha)$ *or* $\mathrm{supp}\, a_{21} \subset (\beta,1)$. *Then, there exists a sequence of real numbers* $(\varepsilon_k)_{k\in\mathbb{N}^*}$ *and* $M > 0$ *such that*

$$\varepsilon_k \xrightarrow[k\to+\infty]{} 0,$$

and

$$\|1_\omega \left(S_k(a_{21}\phi_k - \mathcal{I}_k\phi_k) - \varepsilon_k\phi_k\right)\|_{L^2(0,1)} \leq M\mathcal{I}_k, \quad \forall k \in \mathbb{N}^*.$$

Proof 1/3

Assume that $T < T_0$ and that there exists $C > 0$ such that

$$\|u(0)\|^2_{L^2(0,1)} + \|w(0)\|^2_{L^2(0,1)} \leq C \int_0^T \|1_\omega u(t)\|^2_{L^2(0,1)}\, dt,$$

for every solution to the adjoint system

$$\begin{cases} -\partial_t u - \partial_x^2 u = a_{21}(x) w & \text{in } Q_T, \\ -\partial_t w - \partial_x^2 w = 0 & \text{in } Q_T, \\ u = w = 0 & \text{on } \Sigma_T, \\ u(T) = u^0, \quad w(T) = w^0 & \text{in } \Omega. \end{cases}$$

Assume that $T < T_0$ and that there exists $C > 0$ such that

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Let us define a sequence of initial data by

$$\begin{pmatrix} u_k^0 \\ w_k^0 \end{pmatrix} = \begin{pmatrix} S_k(a_{21}\phi_k - \mathcal{I}_k \phi_k) \\ \phi_k \end{pmatrix} - \varepsilon_k \begin{pmatrix} \phi_k \\ 0 \end{pmatrix}.$$

Since $(u_k^0, w_k^0) \in \ker(-\lambda_k - \mathcal{A}^*)^2$, the corresponding solution (u_k, w_k) is given by

$$\begin{pmatrix} u_k(t) \\ w_k(t) \end{pmatrix} = e^{-k^2\pi^2(T-t)} \left(-(T-t)(-k^2\pi^2 - \mathcal{A}^*) \begin{pmatrix} u_k^0 \\ w_k^0 \end{pmatrix} + \begin{pmatrix} u_k^0 \\ w_k^0 \end{pmatrix} \right).$$

By definition of S_k we have

$$\begin{pmatrix} u_k(t) \\ w_k(t) \end{pmatrix} = e^{-k^2\pi^2(T-t)} \begin{pmatrix} (T-t)\mathcal{I}_k\phi_k + u_0^k \\ w_0^k \end{pmatrix}.$$

Thus,

$$\begin{aligned} \|u_k(0)\|^2_{L^2(0,1)} + \|w_k(0)\|^2_{L^2(0,1)} &= e^{-2k^2\pi^2 T}\Big(\underbrace{T^2\mathcal{I}_k^2 + \|S_k(a_{21}\phi_k - \mathcal{I}_k\phi_k)\|^2_{L^2(0,1)} + \varepsilon_k^2}_{\geq 0} \\ &\quad -T\varepsilon_k\mathcal{I}_k + 1\Big) \\ &\geq e^{-2k^2\pi^2 T}\left(-T\varepsilon_k\mathcal{I}_k + 1\right), \end{aligned}$$

and

$$\begin{aligned} \int_0^T \|1_\omega u_k(t)\|^2_{L^2(0,1)}\,dt &\leq 2\int_0^T e^{-2k^2\pi^2(T-t)}\Big((T-t)^2\|1_\omega\mathcal{I}_k\phi_k\|^2_{L^2(0,1)} \\ &\quad + \|1_\omega\left(S_k(a_{21}\phi_k - \mathcal{I}_k\phi_k) - \varepsilon_k\phi_k\right)\|^2_{L^2(0,1)}\Big)\,dt \\ &\leq 2T\left(T^2 + M^2\right)\mathcal{I}_k^2. \end{aligned}$$

Consequently,

$$1 \leq 2T\left(T^2 + M^2\right)\mathcal{I}_k^2 e^{2k^2\pi^2 T} + T\varepsilon_k\mathcal{I}_k.$$

Proof 3/3

Since $T < T_0$, let $\delta > 0$ be such that $T + \delta < T_0$. Recall that

$$T_0 = \limsup_{k \to +\infty} \frac{\log \frac{1}{|\mathcal{I}_k|}}{k^2 \pi^2}.$$

By definition of lim sup, there exists a subsequence such that

$$T + \delta < \frac{\log \frac{1}{|\mathcal{I}_k|}}{k^2 \pi^2}.$$

As a result

$$|\mathcal{I}_k| \, e^{k^2 \pi^2 T} \leq e^{-k^2 \pi^2 \delta} \xrightarrow[k \to +\infty]{} 0,$$

a contradiction with the previous inequality:

$$1 \leq 2T \left(T^2 + M^2 \right) \mathcal{I}_k^2 e^{2k^2 \pi^2 T} + T \varepsilon_k \mathcal{I}_k.$$

□

BOUNDARY CONTROLLABILITY

Boundary approximate controllability in one dimension

Let us now consider the boundary controllability problem in one-dimension:

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{in } Q_T, \\ \partial_t y_2 - \partial_x^2 y_2 = a_{21}(x) y_1 & \text{in } Q_T, \\ y_1(t,0) = v(t),\ y_2(t,1) = 0, \quad y_2(t,0) = y_2(t,1) = 0 & \text{on } \Sigma_T, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } \Omega. \end{cases} \tag{10}$$

Theorem (G. O. (2014))

System (10) is approximately controllable if, and only if,

$$\mathcal{I}_k \neq 0, \quad \forall k \in \mathbb{N}^*. \tag{11}$$

Theorem (F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa (2014))

Assume that (11) *holds. Let* $T_0 = \limsup_{k \to +\infty} \frac{\log \frac{1}{|\mathcal{I}_k|}}{k^2 \pi^2}$.

1. *If* $T > T_0$, *then System* (10) *is null-controllable at time* T.
2. *If* $T < T_0$, *then System* (10) *is not null-controllable at time* T.

A comparison between distributed and boundary controllability:

Remark

(Syst) *can be approximately controllable while* (10) *can be not approximately controllable.*

Outline

1. Review of the controllability for the heat equation
2. Controllability of a parabolic system
   - Carleman estimates
   - Spectral conditions
   - Geometric control conditions
   - Minimal time of control
   - Boundary controllability
3. Comments

Comments

Summary:

- 1D case: solved (geometric conditions, minimal time).
- ND case: null-controllability if $\omega \cap \operatorname{supp} a_{21} \neq \emptyset$.
- ND case: sufficient conditions for approximate controllability.

Open problems:

- Controllability in dimension $N > 1$.
- More coupled systems.
- Time and space varying couplings.

Comments

Summary:

- 1D case: solved (geometric conditions, minimal time).
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THANK YOU FOR YOUR ATTENTION !