An introduction to linear control theory

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Seminar – Applied Mathematics

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Institut de Mathématiques de Bordeaux

Outline

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1. Introduction

2. Controllability in finite dimension
   - Duality
   - Kalman rank condition, Hautus test
   - Time-dependent systems
   - Control cost

3. Controllability in infinite dimension : the heat equation
   - Introduction
   - Towards systems of heat equations

Regulation of the heat in a room

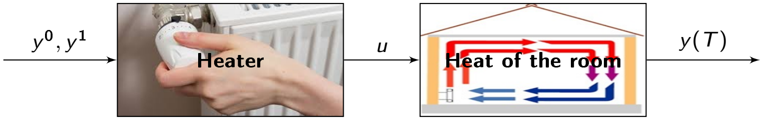
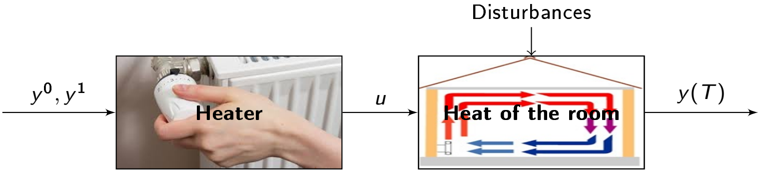


FIGURE – Open-loop control

- y^0 : Initial temperature (e.g. 15°C),
- y^1 : Desired temperature (e.g. 22°C)
- u : Heater (control),
- $y(T)$: Temperature at time T, after the action of the control.



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Drawback : Open-loop controls do not check if the output has achieved the desired goal.

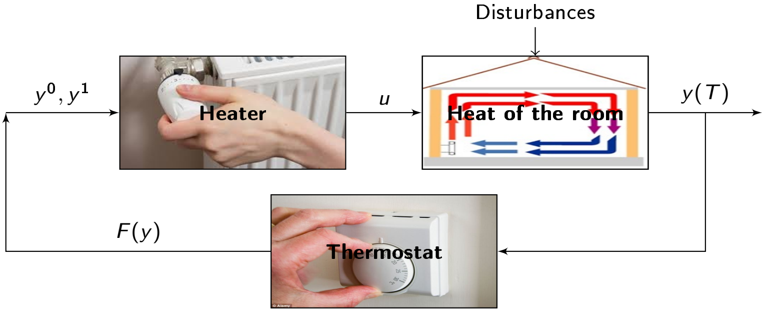

FIGURE – Closed-loop control

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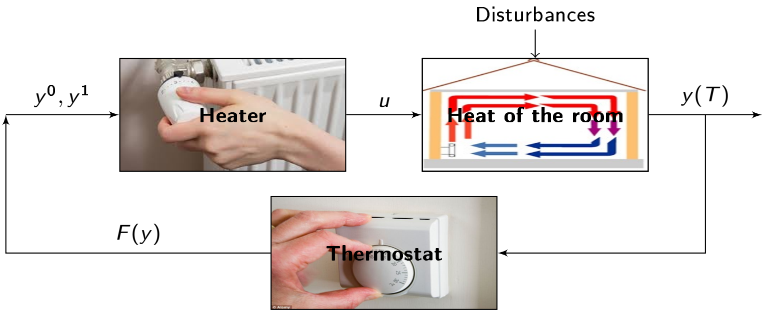
- $F(y)$: Thermostat (feedback).



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In this talk, only open-loop controls !

However – theoretically – closed-loop controls can be obtained by open loop controls ("controllability $\Longrightarrow$ stabilization").

In control theory, we address the following issues :

- Existence of a control
- Existence of controls with various physical constraints
- Estimate the cost of the control
- Design of the control

Confluence of the mathematics

In control theory, we use several fields of Mathematics :

- Differential Equations
- Spectral theory
- Nonlinear analysis
- Functional analysis
- Numerical analysis
- Stochastic processes

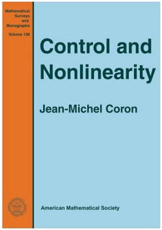

FIGURE – Control and Nonlinearity, J.-M. CORON (2007)

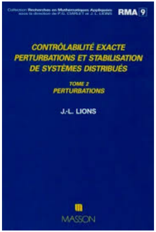

FIGURE – Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, J.-L. LIONS (1988)

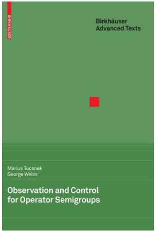

FIGURE – Observation and Control for Operator Semigroups, M. TUCSNAK AND G. WEISS (2009)

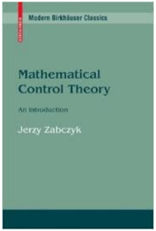

FIGURE – Mathematical Control Theory, J. ZABCZYK (1992)

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System description

Consider

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in (0, T), \\ y(0) = y^0, \end{cases} \tag{ODE}$$

where

- $T > 0$ is the time of control.
- $y = (y_1, \ldots, y_n)$ is the state.
- $y^0 \in \mathbb{R}^n$ is the initial data.
- $A \in \mathbb{R}^{n \times n}$ is a coupling matrix.
- $u \in L^2(0, T)^m$ are the m controls (possibly $m < n$).
- $B \in \mathbb{R}^{n \times m}$ localizes algebraically the controls.

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Well-posedness : For every $y^0 \in \mathbb{R}^n$ and $u \in L^2(0,T)^m$, there exists a unique solution

$$y(t) = e^{tA}y^0 + \int_0^t e^{(t-s)A}Bu(s)\,ds.$$

Note that $y \in C^0([0,T])^n$.

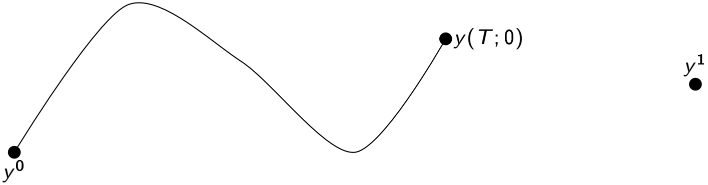

FIGURE – Uncontrolled trajectory

- y^0 : initial state, y^1 : target,
- $y(T;u)$: value of the solution to (ODE) at time T with control u.

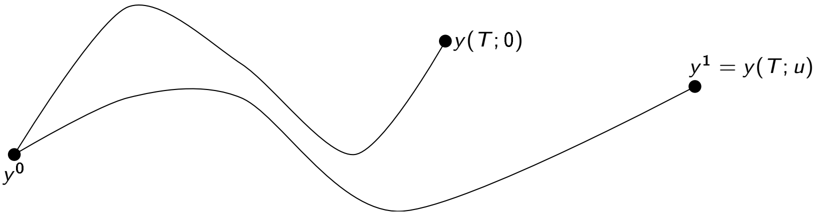

FIGURE – Trajectory controlled exactly

- y^0 : initial state, y^1 : target,
- $y(T; u)$: value of the solution to (ODE) at time T with control u.

Definition

(ODE) is exactly controllable at time T if

$$\forall y^0, y^1 \in \mathbb{R}^n, \exists u \in L^2(0, T)^m, \qquad y(T) = y^1.$$

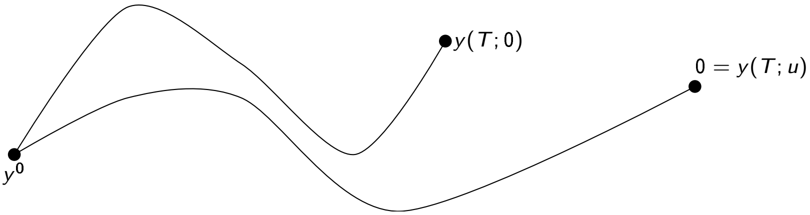

FIGURE – Trajectory controlled to 0

- y^0 : initial state, y^1 : target,
- $y(T; u)$: value of the solution to (ODE) at time T with control u.

Definition

(ODE) is null-controllable at time T if

$$\forall y^0 \in \mathbb{R}^n, \exists u \in L^2(0,T)^m, \qquad y(T) = 0.$$

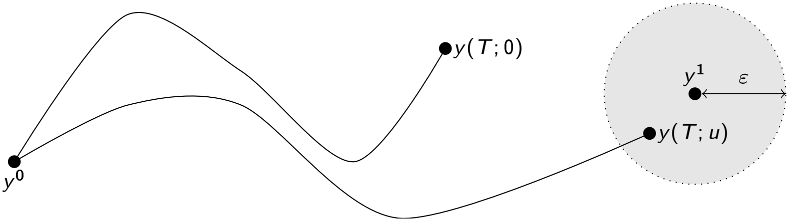

FIGURE – Trajectory controlled approximately

- y^0 : initial state, y^1 : target,
- $y(T;u)$: value of the solution to (ODE) at time T with control u.

Definition

(ODE) is approximately controllable at time T if

$$\forall y^0, y^1 \in \mathbb{R}^n, \forall \varepsilon > 0, \exists u \in L^2(0,T)^m, \qquad \|y(T) - y^1\|_{\mathbb{R}^n} \leq \varepsilon.$$

Reformulation

Let

$$S(T) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad y^0 \longmapsto \overline{y}(T), \qquad \begin{cases} \frac{d}{dt}\overline{y} = A\overline{y}, & t \in (0,T), \\ \overline{y}(0) = y^0, \end{cases}$$

and

$$G_T : L^2(0,T)^m \longrightarrow \mathbb{R}^n, \quad u \longmapsto \widehat{y}(T), \qquad \begin{cases} \frac{d}{dt}\widehat{y} = A\widehat{y} + Bu, & t \in (0,T), \\ \widehat{y}(0) = 0, \end{cases}$$

so that $y(T) = S(T)y^0 + G_T u$.

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so that $y(T) = S(T)y^0 + G_T u$. Therefore,

- (ODE) is exactly controllable at time T if, and only if,

$$\mathrm{Im}\, G_T = \mathbb{R}^n. \tag{1}$$

- (ODE) is null-controllable at time T if, and only if,

$$\mathrm{Im}\, S(T) \subset \mathrm{Im}\, G_T. \tag{2}$$

- (ODE) is approximately controllable at time T if, and only if,

$$\overline{\mathrm{Im}\, G_T} = \mathbb{R}^n. \tag{3}$$

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Remark : All these notions are equivalent.

- (1) $\iff$ (2) since $\mathrm{Im}\, S(T) = \mathbb{R}^n$.
- (1) $\iff$ (3) since $\dim \mathrm{Im}\, G_T < +\infty$.

Duality

Note that $G_T \in \mathcal{L}(L^2(0,T)^m, \mathbb{R}^n)$. Thus,

$$\overline{\mathrm{Im}\, G_T} = \mathbb{R}^n \quad \Longleftrightarrow \quad \ker G_T^* = \{0\}\,. \tag{4}$$

Let us compute G_T^*.

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Let us compute G_T^*. Multiplying (ODE) by z, solution to

$$\begin{cases} -\dfrac{d}{dt} z &=& A^* z, \quad t \in (0,T), \\ z(T) &=& z^1, \end{cases}$$

we obtain

$$y(T) \cdot z^1 - y^0 \cdot z(0) = \int_0^T u(t) \cdot B^* z(t)\, dt.$$

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Using (4), (ODE) is controllable at time T if, and only if,

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Remark : The controllability of (ODE) does not depend on T since $z(t) = e^{(T-t)A^*} z^1$ is analytic.

Proposition (Kalman, Ho and Narendra, 1963)

(ODE) *is controllable if, and only if, the* $n \times n$ *matrix*

$$\Lambda_T = G_T G_T^* \quad \text{is invertible.}$$

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Controllability Gramian - HUM operator

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Remark : We can explicitly compute Λ_T. Indeed,

$$G_T u = \int_0^T e^{(T-t)A} B u(t)\, dt, \qquad G_T^* z^1(t) = B^* e^{(T-t)A^*} z^1,$$

so that

$$\Lambda_T = \int_0^T e^{(T-t)A} B B^* e^{(T-t)A^*}\, dt.$$

Kalman rank condition

$$\begin{cases} \frac{d}{dt}y &= Ay + Bu, \quad t \in (0,T), \\ y(0) &= y^0. \end{cases} \tag{ODE}$$

Theorem (Kalman, Ho and Narendra, 1963)

(ODE) *is controllable if, and only if,*

$$\mathrm{rank}\,(B|AB|\cdots|A^{n-1}B) = n.$$

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Remark : This condition does not depend on T (as expected).
Example : The 2×2 system

$$\begin{cases} \frac{d}{dt}y_1 = \alpha y_1 + \beta y_2 + u, \\ \frac{d}{dt}y_2 = \gamma y_1 + \delta y_2, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, \end{cases} \quad t \in (0,T),$$

is controllable if, and only if,

$$\gamma \neq 0.$$

Proof of the Kalman condition

We have to establish that

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* z(t) = 0, \quad t \in (0,T)\Big) \Longrightarrow z^1 = 0,$$

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Since z is analytic, $B^* z = 0$ if, and only if,

$$\frac{d^k}{dt^k}(B^* z)(T) = 0, \quad \forall k \in \{0, 1, \ldots\},$$

that is,

$$B^* (A^*)^k z^1 = 0, \quad \forall k \in \{0, 1, \ldots\}.$$

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By the Cayley-Hamilton theorem, this is equivalent to

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Thus,

$$z^1 \in \ker \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix} = \ker \left(B | AB | \cdots | A^{n-1} B \right)^* = \left(\mathrm{Im}\ \left(B | AB | \cdots | A^{n-1} B \right) \right)^{\perp},$$

and $\left(\mathrm{Im}\ \left(B | AB | \cdots | A^{n-1} B \right) \right)^{\perp} = (\mathbb{R}^n)^{\perp} = \{0\}$. □

Consider the second order ODE :

$$\begin{cases} \frac{d^2}{dt^2}y = Ay + Bu, & t \in (0,T), \\ y(0) = y^0, & \frac{d}{dt}y(0) = \dot{y}^0. \end{cases} \tag{5}$$

Corollary

(5) *is controllable if, and only if,*

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Proof : 1) Introduce

$$\tilde{y} = \begin{pmatrix} y \\ \frac{d}{dt} y \end{pmatrix} \in \mathbb{R}^{2n}, \quad \tilde{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ A & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \quad \tilde{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathbb{R}^{2n \times m}.$$

2) We have

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = 2n.$$

□

There is a **dual condition** to the Kalman rank condition :

Theorem (Fattorini, 1966 ; Hautus, 1969)

(ODE) *is controllable if, and only if,*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{6}$$

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Taking $t = T$, we see that

$$N \subset \ker B^*.$$

Taking the derivative at time $t = T$, we have

$$A^* N \subset N.$$

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$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{6}$$

Proof : Let

$$N = \left\{z^1 \in \mathbb{R}^n, \quad B^* z(t) = 0, \quad t \in (0, T)\right\}.$$

We have to establish that

$$N = \{0\}.$$

Taking $t = T$, we see that

$$N \subset \ker B^*.$$

Taking the derivative at time $t = T$, we have

$$A^* N \subset N.$$

Thus, if $N \neq \{0\}$, there exist $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{C}^n$ such that

$$\xi \neq 0, \quad \xi \in \ker(\lambda - A^*) \cap \ker B^*,$$

a contradiction with (6). □

Silverman-Meadows rank condition

Let us consider

$$\begin{cases} \frac{d}{dt}y &= A(t)y + B(t)u, \quad t \in (0, T), \\ y(0) &= y^0, \end{cases} \tag{7}$$

with

$$A \in C^\infty([0, T])^{n \times n}, \quad B \in C^\infty([0, T])^{n \times m}.$$

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$$\begin{cases} \frac{d}{dt}y = A(t)y + B(t)u, & t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{7}$$

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Theorem (Silverman and Meadows, 1967)

(7) *is controllable at time T if*

$$\exists \tau \in [0,T], \quad \mathrm{rank}\,(B_0(\tau)|B_1(\tau)|\cdots|B_{n-1}(\tau)) = n, \tag{8}$$

where

$$\begin{cases} B_0(t) = B(t), \\ B_i(t) = -\frac{d}{dt}B_{i-1}(t) + A(t)B_{i-1}(t), \quad \forall i \in \{1,\dots,n-1\}. \end{cases}$$

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Remarks :

- $(B_0(t)|B_1(t)|\cdots|B_{n-1}(t)) = (B|AB|\cdots|A^{n-1}B)$ if $A(t) = A$ and $B(t) = B$.
- (8) is necessary if A and B are analytic.

Assume that (ODE) is controllable.

The control is not unique !

Consider the minimization problem

$$\min_{u \in U_T} \frac{1}{2} \|u\|^2_{L^2(0,T)^m},$$

where

$$U_T = \left\{ u \in L^2(0,T)^m, \quad y(T) = y^1 \right\}.$$

There exists a unique solution $u_{\mathrm{opt}} = \mathrm{proj}_{U_T}(0)$, characterized by

$$\langle u_{\mathrm{opt}}, u_{\mathrm{opt}} - u \rangle_{L^2} \leq 0, \quad \forall u \in U_T. \tag{9}$$

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Proposition (Kalman, Ho and Narendra, 1963)

$$u_{\mathrm{opt}}(t) = B^* e^{(T-t)A^*} \Lambda_T^{-1} \left(y^1 - e^{TA} y^0 \right).$$

Remarks : $u_{\mathrm{opt}} \in C^\infty([0,T])^m$ (u_{opt} is even analytic). Obviously, $y^1 = e^{TA} y^0 \Longrightarrow u_{\mathrm{opt}} = 0$.

Optimal control

Assume that (ODE) is controllable.

The control is not unique !

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Proof : Since $U_T = u_{\mathrm{opt}} + \ker G_T$, (9) gives

$$\langle u_{\mathrm{opt}}, u \rangle_{L^2} = 0, \quad \forall u \in \ker G_T.$$

Thus, $u_{\mathrm{opt}} \in (\ker G_T)^\perp = \overline{\mathrm{Im}\, G_T^*}$: $G_T^* z_n^1 \to u_{\mathrm{opt}}$. But $(z_n^1)_{n \in \mathbb{N}}$ converges ($u_{\mathrm{opt}} \in U_T$) :

$$z_n^1 \xrightarrow[n \to +\infty]{} (G_T G_T^*)^{-1} \left(y^1 - S(T) y^0 \right).$$

□

Cost of controllability

Let $y^0 = 0$. We introduce the control cost

$$C_T = \sup_{\|y^1\|=1} \|u_{\mathrm{opt}}\|_{L^2(0,T)^m}.$$

We have

- C_T is decreasing in T.
- $C_T \to +\infty$ as $T \to 0^+$.

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More precisely,

Theorem (Seidman, 1988)

$$C_T \sim \gamma \frac{1}{T^{K+\frac{1}{2}}} \quad \text{as } T \to 0^+,$$

where $\gamma > 0$ and K is the smallest exponent such that $\mathrm{rank}\,(B|AB|\cdots|A^K B) = n$.

Assume that (ODE) is controllable.

Proposition

One can find controls u ($\neq u_{\mathrm{opt}}$) such that, in addition, given $u^0, u^1 \in \mathbb{R}^m$,

$$u(0) = u^0, \quad u(T) = u^1.$$

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Proof : Consider the $(n+m) \times (n+m)$ system

$$\begin{cases} \frac{d}{dt} y = Ay + Bu, & \\ \frac{d}{dt} u = v, & t \in (0,T), \\ y(0) = y^0, \quad u(0) = u^0. & \end{cases}$$

□

More finite dimensional control theory

Important subjects we did not discuss here :

- Nonlinear control theory
- Stabilization
- Controllability with constraints
- Numerical analysis of controlled systems

Outline

Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ a bounded open subset and set

$$Q_T = (0,T) \times \Omega, \quad \Sigma_T = (0,T) \times \partial\Omega.$$

The heat equation is

$$\begin{cases} \partial_t y = \Delta y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{heat}$$

where

- y is the state, y^0 the initial data,
- $u \in L^2(Q_T)$ is the control,
- $\omega \subset \Omega$ localizes in space the control.
- $\mathbb{1}_\omega(x) = 1$ if $x \in \omega$ and $\mathbb{1}_\omega(x) = 0$ otherwise.

Well-posedness : For every $y^0 \in L^2(\Omega)$ and $u \in L^2(Q_T)$, there exists a unique (weak) solution

$$y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

Remarks

As before, we say that :

- (heat) is exactly controllable at time T if

$$\forall y^0, y^1 \in L^2(\Omega),\ \exists u \in L^2(Q_T), \qquad y(T) = y^1.$$

- (heat) is null-controllable at time T if

$$\forall y^0 \in L^2(\Omega),\ \exists u \in L^2(Q_T), \qquad y(T) = 0.$$

- (heat) is approximately controllable at time T if

$$\forall y^0, y^1 \in L^2(\Omega),\ \forall \varepsilon > 0,\ \exists u \in L^2(Q_T), \qquad \|y(T) - y^1\|_{L^2(\Omega)} \leq \varepsilon.$$

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$$\forall y^0, y^1 \in L^2(\Omega),\ \forall \varepsilon > 0,\ \exists u \in L^2(Q_T), \qquad \|y(T) - y^1\|_{L^2(\Omega)} \leq \varepsilon.$$

Remarks :

- Exact controllability to a state $y^1 \notin C^\infty$ is impossible (regularizing effect in $\Omega \backslash \overline{\omega}$),
- Approximate controllability does not depend on T (analyticity in time of the adjoint system).

Duality

Let us introduce the adjoint system to (heat) :

$$\begin{cases} -\partial_t z = \Delta z & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(T) = z^1 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \partial_t y = \Delta y + \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \qquad \text{(heat)}$$

Multiplying (heat) by z we have

$$\left\langle y(T), z^1 \right\rangle_{L^2} - \left\langle y^0, z(0) \right\rangle_{L^2} = \int_0^T \langle u(t), \mathbb{1}_\omega z(t) \rangle_{L^2} \, dt.$$

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Theorem (Dolecki and Russell, 1977)

- (heat) *is approximately controllable at time* T *if, and only if,*

$$\forall z^1 \in L^2(\Omega), \quad \Big(\mathbb{1}_\omega z(t) = 0, \quad t \in (0, T) \Big) \Longrightarrow z^1 = 0.$$

- (heat) *is null-controllable at time* T *if, and only if,*

$$\exists C_T > 0, \quad \|z(0)\|_{L^2}^2 \leq C_T^2 \int_0^T \|\mathbb{1}_\omega z(t)\|_{L^2}^2 \, dt, \quad \forall z^1 \in L^2(\Omega).$$

C_T is again the control cost !

Theorem

(heat) *is null-controllable at time* T *for any* $T > 0$ *and any nonempty open subset* $\omega \subset \Omega$*.*

Remark : The proof is easy if $\omega = \Omega$, but very complex if $\omega \subsetneq \Omega$.

Theorem

(heat) *is null-controllable at time T for any $T > 0$ and any nonempty open subset $\omega \subset \Omega$.*

Remark : The proof is easy if $\omega = \Omega$, but very complex if $\omega \subsetneq \Omega$.
Proved by :

- Fattorini and Russell (1971), using the method of moments (proof in dimension $N = 1$),
- Lebeau and Robbiano (1995), using elliptic Carleman estimates,
- Fursikov and Imanuvilov (1996), using parabolic Carleman estimates,
- Everdoza and Zuazua (2011), using the transmutation method (from the wave equation to the heat equation), introduced by Miller (2006).

Boundary controllability :

$$\begin{cases} \partial_t y = \Delta y & \text{in } Q_T, \\ y = \mathbb{1}_\gamma u & \text{on } \Sigma_T, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{10}$$

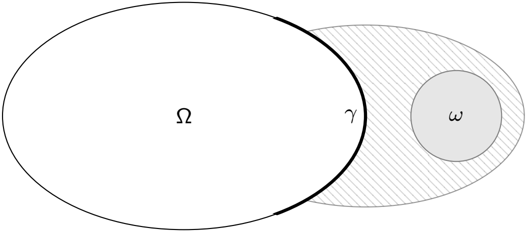
where $\gamma \subset \partial\Omega$ is a nonempty relative open subset.

Boundary controllability :

$$\begin{cases} \partial_t y = \Delta y & \text{in } Q_T, \\ y = \mathbb{1}_\gamma u & \text{on } \Sigma_T, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{10}$$

where $\gamma \subset \partial\Omega$ is a nonempty relative open subset.

Remark : The controllability of (10) is a consequence of the one of (heat) (and vice versa).
1) We extend the domain.



2) We take the trace of the controlled extended solution as control. □

Consider the system of 2 equations with only 1 distributed control :

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + a(x) y_1 & \text{in } Q_T, \\ y_1 = y_2 = 0 & \text{on } \Sigma_T, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \text{in } \Omega, \end{cases} \tag{syst}$$

where

- (y_1, y_2) is the state and $(y_1^0, y_2^0) \in L^2(\Omega)^2$ the initial data,
- $u \in L^2(Q_T)$ is the control,
- $\omega \subset \Omega$ localizes in space the control,
- $a \in C^0(\overline{\Omega})$ is the coupling.

Systems of coupled heat equations

Consider the system of 2 equations with only 1 distributed control :

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- $u \in L^2(Q_T)$ is the control,
- $\omega \subset \Omega$ localizes in space the control,
- $a \in C^0(\overline{\Omega})$ is the coupling.

We can show that :

- If $a(x) = a$ is constant, (syst) is null-controllable at time T if, and only if,

$$a \neq 0.$$

- More generally, (syst) is null-controllable at time T if

$$a \neq 0 \text{ in } \omega.$$

- If $a = 0$ in ω, everything may happen.

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + a(x) y_1 & \text{in } Q_T. \end{cases} \tag{syst}$$

• **Distributed $\neq$ boundary controls :** There exists $a(x)$ such that (syst) is distributed approximately controllable but NOT boundary approximately controllable.

Unexpected behaviors for systems

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + a(x) y_1 & \text{in } Q_T. \end{cases} \qquad \text{(syst)}$$

• **Distributed $\neq$ boundary controls :** There exists $a(x)$ such that (syst) is distributed approximately controllable but NOT boundary approximately controllable.

• **Outbreak of the geometry :**

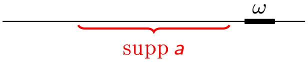

FIGURE – Case n°1 : ω is connected

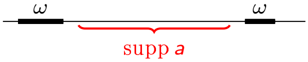

FIGURE – Case n°2 : ω is NOT connected

There exists $a(x)$ such that :

- (syst) is NOT approximately controllable in Case n°1.
- (syst) is approximately controllable in Case n°2.

Unexpected behaviors for systems

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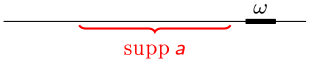
• **Distributed ≠ boundary controls :** There exists $a(x)$ such that (syst) is distributed approximately controllable but NOT boundary approximately controllable.

• **Outbreak of the geometry :**



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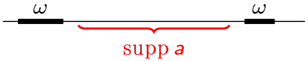

FIGURE – Case n°2 : ω is NOT connected

There exists $a(x)$ such that :

- (syst) is NOT approximately controllable in Case n°1.
- (syst) is approximately controllable in Case n°2.

• **Minimal time of control :** There exists $a(x)$ such that, for some $T^* > 0$,

- For every $T > T^*$, (syst) is null-controllable at time T.
- For every $T < T^*$, (syst) is NOT null-controllable at time T.

Unexpected behaviors for systems

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 + a(x) y_1 & \text{in } Q_T. \end{cases} \qquad \text{(syst)}$$

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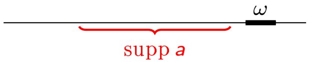

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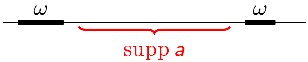

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There exists $a(x)$ such that :

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- For every $T > T^*$, (syst) is null-controllable at time T.
- For every $T < T^*$, (syst) is NOT null-controllable at time T.

• **Null $\neq$ approximate controllability :** There exists $a(x)$ such that (syst) is approximately controllable but NOT null-controllable at any time T.

- ***Insensitizing controls for a semilinear heat equation,*** L. de Teresa, Comm. Partial Differential Equations, 25 (2000), 39-72.
- ***Boundary controllability of parabolic coupled equations,*** E. Fernàndez-Cara, M. Gonzàlez-Burgos and L. de Teresa, J. Funct.Anal., 259 (2010), 1720-1758.
- ***Boundary approximate controllability of some linear parabolic systems,*** G. Olive, Evol. Equ. Control Theory 3 (2014), no. 1, 167-189.
- ***Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients,*** F. Boyer and G. Olive, Math. Control Relat. Fields 4 (2014), no. 3, 263-287.
- ***Minimal time for the null controllability of parabolic systems : The effect of the condensation index of complex sequences,*** F. Ammar-Khodja, A. Benabdallah, M. Gonzàlez-Burgos, and L. de Teresa, J. Funct. Anal., 267 (2014), pp. 2077–2151.

Dziękuję za uwagę !