Math. Control Signals Syst. (2012) 23:257–280
DOI 10.1007/s00498-011-0071-x

ORIGINAL ARTICLE

Null-controllability for some linear parabolic systems with controls acting on different parts of the domain and its boundary

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Received: 26 June 2011 / Accepted: 12 December 2011 / Published online: 23 December 2011

Abstract In this work, we study the null-controllability properties of linear parabolic systems with constant coefficients in the case where several controls are acting on different distributed subdomains and/or on the boundary. We prove a Kalman rank condition in the one-dimensional case. In the case where only distributed controls are considered, we also establish related results such as a Carleman estimate.

Keywords Kalman rank condition · Boundary controllability · Distributed controllability · Carleman estimate

1 Introduction

Let $n \in \mathbb{N}^*$, $n_D, n_B \in \mathbb{N}$, be respectively the number of equations, the number of distributed controls and the number of boundary controls that we will consider. Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}^*$) be a bounded connected open set with boundary $\partial\Omega$ regular enough. For every $T > 0$, we denote $Q_T = (0, T) \times \Omega$ and $\Sigma_T = (0, T) \times \partial\Omega$. Let $\omega_1, \ldots, \omega_{n_D}$ be given non-empty open subsets of Ω (possibly disjoint) and let $\Gamma_1, \ldots, \Gamma_{n_B}$ be given non-empty open subsets of $\partial\Omega$. We consider the following type of $n \times n$ parabolic system:

$$\begin{cases} \partial_t y = \Delta y + Ay + D_1 u_1(t,x) 1_{\omega_1}(x) + \cdots + D_{n_D} u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y = B_1 v_1(t,x) 1_{\Gamma_1}(x) + \cdots + B_{n_B} v_{n_B}(t,x) 1_{\Gamma_{n_B}}(x) \text{ on } \Sigma_T, \end{cases} \quad (1)$$

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where y is the state, $A \in \mathcal{M}_n(\mathbb{R})$ is a coupling matrix. For every $i \in \{1, \dots, n_D\}$, $D_i \in \mathbb{R}^n$ and $u_i \in L^2(Q_T)$ is a *distributed control* acting on ω_i. For every $j \in \{1, \dots, n_B\}$, $B_j \in \mathbb{R}^n$ and $v_j \in L^2(\Sigma_T)$ is a *boundary control* acting on Γ_j.

Let us recall that, for every $T > 0$, for every $y_0 \in L^2(\Omega; \mathbb{R}^n)$, $u_i \in L^2(Q_T)$, $i \in \{1, \dots, n_D\}$, and $v_j \in L^2(\Sigma_T)$, $j \in \{1, \dots, n_B\}$, there exists a unique solution $y \in L^2(Q_T; \mathbb{R}^n) \cap C^0([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$ to (1), defined by transposition, which satisfies $y(0) = y_0$ (see for instance [8, Appendix] for more details).

Let be given $y_0 \in L^2(\Omega; \mathbb{R}^n)$ and $T > 0$, it will be said that system (1) is *null-controllable on* $(0, T)$ *from the state* y_0 if there exists $u_i \in L^2(Q_T)$ for every $i \in \{1, \dots, n_D\}$ and there exists $v_j \in L^2(\Sigma_T)$ for every $j \in \{1, \dots, n_B\}$ such that the corresponding solution to (1) with $y(0) = y_0$ satisfies $y(T) = 0$. Let be given $T > 0$, it will be said that system (1) is *null-controllable on* $(0, T)$ if for every $y_0 \in L^2(\Omega; \mathbb{R}^n)$ system (1) is null-controllable on $(0, T)$ from the state y_0. It will be said that system (1) is *null-controllable* if for every $T > 0$ system (1) is null-controllable on $(0, T)$.

Let us recall that the scalar operator $-\Delta$ with homogeneous Dirichlet boundary condition admits a sequence of eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*} \subset \mathbb{R}^*_+$ such that the associated sequence of normalized eigenfunctions $\{\phi_k\}_{k \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\Omega)$.

Matrices notation *For any $q, N_1, N_2 \in \mathbb{N}^*$, for any matrix $A \in \mathcal{M}_{N_1}(\mathbb{R})$, $B \in \mathcal{M}_{N_1 \times N_2}(\mathbb{R})$, we denote $[A|B]$ the matrix whose first columns are those of A and the following ones are those of B and we define*

$$
[\, A \, : \, B \,]_q = \left[B | AB | \cdots | A^{q-1} B \right] \in \mathcal{M}_{N_1 \times N_2 q}(\mathbb{R}),
$$

$$
((B))_q = \begin{bmatrix} B & 0 & \cdots & \cdots & 0 \\ 0 & B & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & B \end{bmatrix} \in \mathcal{M}_{N_1 q \times N_2 q}(\mathbb{R}),
$$

$$
(B)_q = \begin{bmatrix} B \\ B \\ \vdots \\ \vdots \\ B \end{bmatrix} \in \mathcal{M}_{N_1 q \times N_2}(\mathbb{R}), \quad (B)^{\pm}_q = \begin{bmatrix} -B \\ B \\ \vdots \\ \vdots \\ (-1)^q B \end{bmatrix} \in \mathcal{M}_{N_1 q \times N_2}(\mathbb{R}), \tag{2}
$$

$$
\mathcal{A}_q = \begin{bmatrix} A_1 & 0 & \cdots & \cdots & 0 \\ 0 & A_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A_q \end{bmatrix} \in \mathcal{M}_{N_1 q}(\mathbb{R}) \text{ with } A_k = -\lambda_k I + A \in \mathcal{M}_{N_1}(\mathbb{R}).
$$

Note that

$$\text{rank}\,(([\,A\,:\,B\,]_n))_q = q\,\text{rank}\,[\,A\,:\,B\,]_n, \quad \forall q \in \mathbb{N}^*. \tag{3}$$

In controllability theory of linear ordinary differential systems, there exists a complete characterization of controllability, this is the so-called *Kalman rank condition* (see for instance [13, Corollary 1.4.10]), that is to say, if $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ ($n, m \in \mathbb{N}^*$), then the linear ordinary differential system $y' = Ay + Bu$ is controllable if and only if

$$\text{rank}\,[\,A\,:\,B\,]_n = n.$$

To give an appropriate condition of this type in the framework of linear parabolic systems has been a subject of several research. Recently in [3], a Kalman rank condition has been proved for the distributed null-controllability with only one control region: the authors proved that the system

$$\begin{cases} \partial_t y = \Delta y + Ay + Du_1(t,x)1_{\omega_1}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases}$$

is null-controllable if and only if

$$\text{rank}\,[\,A\,:\,D\,]_n = n, \tag{4}$$

see [3, Theorem 1.1] and [3, Proposition 2.2]. In fact, they proved a more general Kalman rank condition for linear parabolic systems with different coefficients in front of the operator $-\Delta$, for more details see [3].

In [8], the authors gave a necessary and sufficient condition for the boundary null-controllability in the one-dimensional case and for two equations (see [8, Theorem 1.1]). Through this theorem, they also showed that the Kalman rank condition for distributed null-controllability is a necessary condition for the boundary null-controllability but it is not sufficient. The result of [8] has been improved in [6] where the authors proved a new Kalman rank condition for boundary null-controllability of $n \times n$ linear parabolic system (still in the one-dimensional case though), that is: the system

$$\begin{cases} \partial_t y = \partial_{xx}^2 y + Ay \text{ in } (0,T)\times(0,1), \\ y(t,0) = B_1 v_1(t), \quad y(t,1) = B_2 v_2(t) \text{ on } (0,T), \end{cases}$$

is null-controllable if and only if

$$\text{rank}\left[\,\mathcal{A}_q\,:\,\left((B_1)_q\middle|(B_2)_q^{\pm}\right)\,\right]_{nq} = nq, \quad \forall q \in \mathbb{N}^*, \tag{5}$$

see [6, Theorem 6.3].

We also point out reference [4] where the authors worked on the case of regular time-dependent matrices $A = A(t)$ and $D = D(t)$ (with one distributed control) and they proved that the system

$$\begin{cases} \partial_t y = \Delta y + A(t)y + D(t)u_1(t,x)1_{\omega_1}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases}$$

is null-controllable if

$$\exists t_0 \in [0,T], \quad \operatorname{rank} \mathcal{K}(t_0) = n, \tag{6}$$

where

$$\mathcal{K}(t) = \left[\mathcal{K}_0(t)|\cdots|\mathcal{K}_{n-1}(t)\right]$$

with

$$\begin{cases} \mathcal{K}_0(t) = D(t) \\ \mathcal{K}_i(t) = A(t)\mathcal{K}_{i-1}(t) - \dfrac{d}{dt}\mathcal{K}_{i-1}(t), \quad \forall i \in \{1,\dots,n-1\} \end{cases}$$

see [4, Theorem 1.2].

Finally, let us mention that the null-controllability properties of linear parabolic systems have been also studied in the case of space varying coefficients and one distributed control force. However few results are known, even for the distributed controllability. Sufficient conditions to the distributed null-controllability are given in [2,5,7,9,11] for systems of two equations and see [10] for $n \times n$ systems. To our knowledge, [7] is also the first result using Carleman estimates for two coupled parabolic equations. Concerning the boundary null-controllability of parabolic systems, let us mention [1] where the authors prove a result without any restriction on the dimension (but under some geometric condition, see [1] for more details).

In the present work, we try to give an overview of the controllability properties of systems like (1). One of the main task of this work will be to prove a Kalman rank condition for system (1) which will then generalize the previously known Kalman conditions (4) and (5).

2 Statements of the results

2.1 Main result

The first and main result of this work concerns system (1) in the case $N = 1$. As a consequence, it is equivalent to consider the following system:

$$\begin{cases} \partial_t y = \partial_{xx}^2 y + Ay + D_1 u_1(t,x)1_{\omega_1}(x) + \cdots + D_{n_D} u_{n_D}(t,x)1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y(t,0) = B_1^L v_1(t) + \cdots + B_{n_L}^L v_{n_L}(t), \quad y(t,1) = B_1^R w_1(t) + \cdots + B_{n_R}^R w_{n_R}(t) \\ \text{on } (0,T), \end{cases} \tag{7}$$

with $n_B \leq n_L + n_R \leq 2n_B$, and where we take $\Omega = (0,1)$ for the sake of simplicity. Let us denote $D = \left[D_1|D_2|\cdots|D_{n_D}\right] \in \mathcal{M}_{n\times n_D}(\mathbb{R})$, $B^L = \left[B_1^L|\cdots|B_{n_L}^L\right]$ and $B^R = \left[B_1^R|\cdots|B_{n_R}^R\right]$. Then, the result reads:

Theorem 1 (Kalman rank condition) *System* (7) *is null-controllable if and only if*

$$\operatorname{rank}\left[\ \left[\mathcal{A}_q : \left(\left(B^L\right)_q \middle| \left(B^R\right)_q^{\pm}\right)\right]_{nq}\ \middle|\ \left(([A:D]_n)\right)_q\ \right] = nq, \quad \forall q \in \mathbb{N}^*, \tag{8}$$

(where we used the notations introduced above).

Remark 1 1. Theorem 1 contains both Kalman condition (4) for distributed null-controllability and Kalman condition (5) for boundary null-controllability, see (3).

2. We can also reformulate condition (8) as follows:

$$\operatorname{rank}\left[\mathcal{A}_q : \left(\left(B^L\right)_q \middle| \left(B^R\right)_q^{\pm} \middle| ((D))_q\right)\right]_{nq} = nq, \quad \forall q \in \mathbb{N}^*. \tag{9}$$

   This is due to the equalities $\operatorname{rank}\left[\mathcal{A}_q : ((D))_q\right]_{nq} = \operatorname{rank}\,(([A : D]_n))_q$ for all $q \in \mathbb{N}^*$. This characterization will be used to prove Theorem 1.

3. Let us observe that condition (8) is only algebraic. In particular, it does not depend on $\omega_1, \ldots, \omega_{n_D}$.
4. Condition (8) can be checked thanks to the following fact: to check condition (8) is equivalent to check it for a particular $q = q_0$ which is such that

$$\mu_i - \mu_j \neq \lambda_k - \lambda_l, \quad \forall k, l \in \mathbb{N}^* \text{ with } k > q_0 \text{ and } l \neq k, \quad \forall i, j \in \{1, \ldots, n\}, \tag{10}$$

   where $\{\mu_k\}_{k\in\{1,\ldots,n\}} \subset \mathbb{C}$ is the set of the eigenvalues of A. To prove this fact, one can adapt the proof of [6, Corollary 3.3], using the characterization (9). Moreover, one can see that such a q_0 does always exist, see [6, Proposition 3.2] for instance.

Let us illustrate the last item of Remark 1 through the following example:

Example 1 Let $T > 0$ and $\omega \subset (0, 1)$ a non-empty open subset. Consider the following 3×3 one-dimensional parabolic system:

$$\begin{cases} \left.\begin{array}{l} \partial_t y_1 = \partial_{xx}^2 y_1 + 2y_1 + 6y_2 + 2y_3, \\ \partial_t y_2 = \partial_{xx}^2 y_2 + 4y_1 - 2y_3, \\ \partial_t y_3 = \partial_{xx}^2 y_3 + 2y_1 - 3/2 y_2 + 2y_3 + u(t,x)1_\omega(x), \end{array}\right\} \text{ in } (0,T)\times(0,1), \\ \left.\begin{array}{ll} y_1(t,0) = v(t), & y_1(t,1) = 0, \\ y_2(t,0) = 0 & y_2(t,1) = 0, \\ y_3(t,0) = 0 & y_3(t,1) = 0, \end{array}\right\} \text{ on } (0,T), \end{cases} \tag{11}$$

so that

$$A = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 0 & -2 \\ 2 & -3/2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_L = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_R = 0.$$

We can see that if only one control is acting then this system is not null-controllable. Indeed we have $\operatorname{rank}[\,A : D\,]_3 = 2 \neq 3$ and $\operatorname{rank}[\,A : B\,]_3 = 2 \neq 3$ so the distributed and boundary Kalman conditions fail. Nevertheless, we have $\operatorname{rank}[\,[\,A : B\,]_3 \,|\, [\,A : D\,]_3\,] = 3$ and the eigenvalues of A are $-5, 3$ and 6 so that condition (10) is satisfied for $q_0 = 1$ and thus, by the previous remark, condition (8) is also satisfied.

We make the following remark about this example, this gives a good idea of the proof of Theorem 1:

Remark 2 Observe that in fact the matrix A of Example 1 is equivalent to the matrix

$$C = \begin{bmatrix} 6 & 0 & 0 \\ 6 & 0 & 15 \\ -2 & 1 & -2 \end{bmatrix}$$

through the following change of basis:

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} = [B|D|AD] \quad \text{(with the notations of Example 1)}\,.$$

As a consequence the null-controllability of system (11) is equivalent to the null-controllability of the system

$$\begin{cases} \partial_t z = \partial_{xx}^2 z + \begin{bmatrix} 6 & 0 & 0 \\ 6 & 0 & 15 \\ -2 & 1 & -2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t,x) 1_\omega(x) \text{ in } (0,T) \times (0,1), \\ z(t,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v(t), \quad z(t,1) = 0 \text{ on } (0,T), \end{cases}$$

And we can see that we can lead the first component of this system to zero at time $T/2$ (for instance) using only the boundary control v. Then, it remains to prove that the system

$$\begin{cases} \partial_t \hat{z} = \partial_{xx}^2 \hat{z} + \begin{bmatrix} 0 & 15 \\ 1 & -2 \end{bmatrix} \hat{z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t,x) 1_\omega(x) \text{ in } \left(\frac{T}{2}, T\right) \times (0,1), \\ \hat{z}(t,0) = 0, \quad \hat{z}(t,1) = 0 \text{ on } \left(\frac{T}{2}, T\right), \end{cases}$$

is null-controllable, which can be done by checking the distributed Kalman condition.

2.2 More results in the case $n_B = 0$

Let us now consider the case without boundary control with an arbitrary space dimension N, that is

$$\begin{cases} \partial_t y = \Delta y + Ay + D_1 u_1(t,x) 1_{\omega_1}(x) + \cdots + D_{n_D} u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T. \end{cases} \tag{12}$$

From the proof of Theorem 1 we will see that in fact Theorem 1 still holds without any restriction on N if we have no boundary controls:

Corollary 1 *Let $N \geq 1$ be arbitrary. Then, system (12) is null-controllable if and only if*

$$\operatorname{rank}\,[\, A \;:\; D\,]_n = n. \tag{13}$$

On the other hand, when the Kalman condition (13) is not fulfilled it is possible to characterize the states that can be driven to 0:

Proposition 1 *Assume that $N \geq 1$ and $\operatorname{rank}\,[\, A : D\,]_n < n$. Then, system (12) is null-controllable on $(0, T)$ from the state y_0 for every $T > 0$ if and only if*

$$y_0 \in L^2\left(\Omega; \operatorname{span}\,[\, A \;:\; D\,]_n\right).$$

This can be proved by extending the arguments given in [3, Theorem 1.5].

We also can also extend the Kalman rank condition for time-dependent matrices (6): let us consider the system

$$\begin{cases} \partial_t y = \Delta y + A(t)y + D_1(t) u_1(t,x) 1_{\omega_1}(x) + \cdots + D_{n_D}(t) u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \\ \qquad\qquad \text{in } Q_T, \\ y = 0 \text{ on } \Sigma_T. \end{cases} \tag{14}$$

where $A \in \mathcal{C}^{n-1}([0,T]; \mathcal{M}_n(\mathbb{R}))$ and $D_i \in \mathcal{C}^n([0,T]; \mathbb{R}^n)$.

In this case, we have

Theorem 2 *If there exists $t_0 \in [0, T]$ such that*

$$\exists t_0 \in [0, T], \quad \operatorname{rank} \mathcal{K}(t_0) = n, \tag{15}$$

where

$$\mathcal{K}(t) = \left[\mathcal{K}_0(t)|\cdots|\mathcal{K}_{n-1}(t)\right]$$

with

$$\begin{cases} \mathcal{K}_0(t) = D(t) = \left[D_1(t)\middle|\cdots\middle|D_{n_D}(t)\right] \\ \mathcal{K}_i(t) = A(t)\mathcal{K}_{i-1}(t) - \dfrac{d}{dt}\mathcal{K}_{i-1}(t), \quad \forall i \in \{1, \ldots, n-1\} \end{cases}$$

then the system (14) is null-controllable at time T.

This theorem will be proved thanks to a Carleman estimate for cascade systems, see Theorem 3 below.

3 The Kalman rank condition

Recall that throughout this section, we assume that $N = 1$ and the system considered is (7). In fact, for the sake of simplicity of the notations we will consider

$$\begin{cases} \partial_t y = \partial_{xx}^2 y + Ay + D_1 u_1(t,x) 1_{\omega_1}(x) + \cdots + D_{n_D} u_{n_D}(t,x) 1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y(t,0) = B_1 v_1(t), \quad y(t,1) = B_2 v_2(t) \text{ on } (0,T). \end{cases} \tag{16}$$

3.1 Some known results

Before starting the proof of Theorem 1 let us recall for convenience the following results:

Proposition 2 *Let be given $A \in \mathcal{M}_n(\mathbb{R})$ and $B \in \mathcal{M}_{n\times m}(\mathbb{R})$ $(n, m \in \mathbb{N}^*)$. We have*

$$\forall V \in \mathbb{R}^n, \quad \left(\forall t \geq 0, \quad B^* e^{tA^*} V = 0\right) \Longrightarrow V = 0$$

if and only if

$$\operatorname{rank}\,[\,A \;:\; B\,]_n = n$$

if and only if

$$\ker(A^* - \theta I) \cap \ker B^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{17}$$

Condition (17) is the so-called *Hautus test*. For a proof see for instance [13, Chapter 1]. To state the second result we need to define the *adjoint system* of (16):

$$\begin{cases} -\partial_t \Phi = \partial_{xx}^2 \Phi + A^* \Phi \text{ in } Q_T, \\ \Phi(t,0) = 0, \quad \Phi(t,1) = 0 \text{ on } (0,T). \end{cases} \tag{18}$$

The introduction of system (18) is of interest thanks to the following proposition, which gives a characterization of the null-controllability of system (16) through an inequality on its adjoint system (18):

Proposition 3 (Observability inequality) *Let be given $T > 0$. System (16) is null-controllable on $(0, T)$ if and only if there exists $\mathbf{C} > 0$ such that for every $\Phi^T \in H_0^1(\Omega; \mathbb{R}^n)$ the solution Φ to the adjoint system (18) with $\Phi(T) = \Phi^T$ satisfies*

$$\|\Phi(0)\|^2_{H^1_0(\Omega;\mathbb{R}^n)} \leq \mathbf{C}\left(\sum_{i=1}^{n_D}\int_0^T\int_{\omega_i}\left|D_i^*\Phi(t,x)\right|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\left|B_1^*\partial_x\Phi(t,0)\right|^2\mathrm{d}t + \int_0^T\left|B_2^*\partial_x\Phi(t,1)\right|^2\mathrm{d}t\right), \quad (19)$$

For a proof see for instance [8, Appendix]. Inequality (19) is called *observability inequality*.

3.2 Proof of Theorem 1

The key point of the proof is to do an appropriate change of basis thanks to the hypothesis (8). In this new basis, the matrix A becomes a block upper triangular matrix C; and B_1, B_2 and D become such that one control is acting on each diagonal block of C. Note that this technique, firstly used in [3], is very specific to the fact that the coefficients before the operator $-\partial^2_{xx}$ are the same on every single equation. In a second time, we will check that every diagonal block of C satisfies the appropriate Kalman condition (boundary or distributed). And as a consequence, taking also advantage of the fact that the last block of C is decoupled from the upper ones, we can start to control the last block in a time before T and lead to zero at this time the components associated to this block; this allows us to iterate the process for the remaining blocks and finally lead every component to zero at time T.

Proof **Step 1** Under the condition (8), we start to construct a basis in which the matrices A, D and B_1, B_2 has the desired structure. We have

Lemma 1 *Assume that condition* (8) *holds. Then, there exists* $r_D \in \{0,\ldots,n_D\}$, $D_{i_1},\ldots,D_{i_{r_D}} \in \{D_k\}_{1\leq k\leq n_D}$ *and* $s_1,\ldots,s_{r_D} \in \{1,\ldots,n\}$ *such that for every* $q \in \mathbb{N}^*$ *there exists* $r_B \in \{0,1,2\}$, $\mathsf{B}_{\mathsf{j}_1},\ldots,\mathsf{B}_{\mathsf{j}_{\mathsf{r}_\mathsf{B}}} \in \left\{(B_1)_q,(B_2)^{\pm}_q\right\}$ *and* $\tilde{s}_1,\ldots\tilde{s}_{r_B} \in \{1,\ldots,nq\}$, *such that*

$$P_q = \left[\, P_q^D \,\middle|\, P_q^B \,\right] \in \mathcal{M}_{nq}(\mathbb{R})$$

is invertible, where we have denoted

$$P_q^D = \left(\left(\left[\left[\,A : D_{i_1}\,\right]_{s_1}\middle|\cdots\middle|\left[\,A : D_{i_{r_D}}\,\right]_{s_{r_D}}\right]\right)\right)_q \text{ and}$$

$$P_q^B = \left[\left[\,\mathcal{A}_q : \mathsf{B}_{\mathsf{j}_1}\,\right]_{\tilde{s}_1}\middle|\cdots\middle|\left[\,\mathcal{A}_q : \mathsf{B}_{\mathsf{j}_{\mathsf{r}_\mathsf{B}}}\,\right]_{\tilde{s}_{r_B}}\right].$$

Moreover for every $k \in \{1,\ldots,r_D\}$,

$$A^{s_k}D_{i_k} \in \operatorname{span}\left[\left[\,A : D_{i_1}\,\right]_{s_1}|\cdots|\left[\,A : D_{i_k}\,\right]_{s_k}\right].$$

Proof **Step 1** We assume that $D \neq 0$ otherwise the result stated by Theorem 1 is already known (see [6]). Thus there exists $D_{i_1} \in \{D_k\}_{1\le k\le n_D}$ such that $D_{i_1} \neq 0$. We set

$$s_1 = \text{rank}\left(D_{i_1}, AD_{i_1}, \ldots, A^{n-1}D_{i_1}\right)$$

so that rank $\left(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}\right) = s_1$. If $s_1 = n$ then the proof ends here by taking $P_q = \left(\left(\left[D_{i_1}|AD_{i_1}|\cdots|A^{s_1-1}D_{i_1}\right]\right)\right)_q$. If $s_1 < n$ we check if there exists $D_{i_2} \in \{D_k\}_{1\le k\le n_D}\setminus\{D_{i_1}\}$ such that $(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}, D_{i_2})$ is linearly independent. If this is not the case we go to step 2. But if such a D_{i_2} exists we set

$$s_2 = \text{rank}\left(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \ldots, A^{n-1}D_{i_2}\right) - s_1$$

so that rank $\left(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \ldots, A^{s_2-1}D_{i_2}\right) = s_1 + s_2$. If $s_1 + s_2 = n$ the proof ends. If $s_1 + s_2 < n$ then we continue the previous process. This stops when we have found a rank $r_D \in \{1, \ldots, n\}$, $i_1, \ldots, i_{r_D} \in \{1, \ldots, n_D\}$ and $s_1, \ldots, s_{r_D} \in \{1, \ldots, n\}$ such that

$$\left(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \ldots, A^{s_2-1}D_{i_2}, \ldots, A^{s_{r_D}-1}D_{i_{r_D}}\right) \tag{20}$$

is linearly independent and such that every element of $\{D_k\}_{1\le k\le n_D}\setminus\{D_{i_1}, \ldots, D_{i_{r_D}}\}$ belongs to the space spanned by the family (20). As said before if $\sum_{k=1}^{r_D} s_k = n$ the proof ends (and let us remark that in this case system (16) is null-controllable with distributed controls alone). If this is not the case:

Step 2 Thanks to condition (8) there exists $\mathsf{B}_{\mathsf{j}_1} \in \left\{(B_1)_q, (B_2)_q^{\pm}\right\}$ and $\hat{s} \in \{1, \ldots, nq\}$ such that

$$\left(\left(\left(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \ldots, A^{s_2-1}D_{i_2}, \ldots, A^{s_{r_D}-1}D_{i_{r_D}}\right)\right)_q,\right.$$
$$\left.\mathcal{A}_q^{\hat{s}-1}\mathsf{B}_{\mathsf{j}_1}\right)$$

is linearly independent. One can check that this necessary implies that the family

$$\left(\left(D_{i_1}, AD_{i_1}, \ldots, A^{s_1-1}D_{i_1}, D_{i_2}, AD_{i_2}, \ldots, A^{s_2-1}D_{i_2}, \ldots, A^{s_{r_D}-1}D_{i_{r_D}}\right)_q, \mathsf{B}_{\mathsf{j}_1}\right)$$

is also linearly independent. We set

$$\tilde{s}_1 = \text{rank}\left(\left(\left(D_{i_1}, \ldots, A^{s_1-1}D_{i_1}, \ldots, D_{i_{r_D}}, \ldots, A^{s_{r_D}-1}D_{i_{r_D}}\right)\right)_q, \mathsf{B}_{\mathsf{j}_1}, \ldots, \mathcal{A}_q^{nq-1}\mathsf{B}_{\mathsf{j}_1}\right)$$
$$-\sum_{k=1}^{r_D} s_k q.$$

If $\tilde{s}_1 + \sum_{k=1}^{r_D} s_k q = nq$ we have done. If this is not the case, thanks to condition (8) we can find $\mathsf{B}_{j_2} \in \left\{(B_1)_q, (B_2)_q^{\pm}\right\} \setminus \left\{\mathsf{B}_{j_1}\right\}$ such that

$$\left(\left(\left(D_{i_1}, \ldots, A^{s_1-1} D_{i_1}, \ldots, D_{i_{r_D}}, \ldots, A^{s_{r_D}-1} D_{i_{r_D}}\right)\right)_q, \mathsf{B}_{j_1}, \ldots, \mathcal{A}_q^{\tilde{s}_1-1}\mathsf{B}_{j_1}, \mathsf{B}_{j_2}\right)$$

is linearly independent. We iterate the same process and finally obtain the result. □

We apply Lemma 1 and for the sake of simplicity of the notations we will treat one case, that is $r_B = 2$ and $\mathsf{B}_{j_1} = (B_1)_q$, $\mathsf{B}_{j_2} = (B_2)_q^{\pm}$. For $q = 1$ we obtain that $P_1 \in \mathcal{M}_n(\mathbb{R})$ is invertible and

$$P_1^{-1} A P_1 = C = \begin{bmatrix} C_1 & \times & \cdots & \cdots & \times \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & C_{r_D} & \times \\ 0 & \cdots & \cdots & 0 & K \end{bmatrix} \text{ where } C_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & \times \\ 1 & \ddots & & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \times \end{bmatrix} \in \mathcal{M}_{s_i}(\mathbb{R})$$

and $K \in \mathcal{M}_{\tilde{s}_B}(\mathbb{R})$ with $\tilde{s}_B = \sum_{k=1}^{2} \tilde{s}_k$. Moreover for every $l \in \{1, \ldots, r_D\}$ we have

$$P_1 e_{S_l} = D_{i_l} \text{ and } P_1 e_{\tilde{S}_1} = B_1, \quad P_2 e_{\tilde{S}_2} = -B_2$$

where we denote $S_k = 1 + \sum_{r=1}^{k-1} s_r$, $\tilde{S}_l = S_{r_D+1} + \sum_{r=1}^{l-1} \tilde{s}_r$ and the vector e_j denotes the real vector of $\mathbb{R}^n$ with 1 on its jth component and 0 elsewhere.

Let us now remark that if the system

$$\begin{cases} \partial_t z = \partial_{xx}^2 z + Cz + e_{S_1}\hat{u}_{i_1}(t,x)1_{\omega_{i_1}}(x) + \cdots + e_{S_{r_D}}\hat{u}_{i_{r_D}}(t,x)1_{\omega_{i_{r_D}}}(x) \text{ in } Q_T, \\ z(t,0) = e_{\tilde{S}_1}\hat{v}_1(t), \quad z(t,1) = -e_{\tilde{S}_2}\hat{v}_2(t) \text{ on } (0,T), \end{cases} \tag{21}$$

is null-controllable, then system (16) is also null-controllable by doing the change of variables $z = P_1^{-1} y$ and then taking for all $l \in \{1, \ldots, n_D\}$ $u_l = \hat{u}_{i_k}$ if there exists $k \in \{1, \ldots, r_D\}$ such that $l = i_k$, $u_l = 0$ otherwise, and taking for all $l \in \{1, 2\}$ $v_l = \hat{v}_l$. So let us now prove that the system (21) is null-controllable:

Step 2 We rewrite the solution z of system (21) as follow:

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_{r_D} \\ z_B \end{bmatrix}$$

where $z_B \in \mathbb{R}^{\tilde{s}_B}$ and $z_i \in \mathbb{R}^{s_i}$ for all $i \in \{1, \ldots, r_D\}$.

Now we look at the system satisfied by z_B and observe that it is independent of $z_1, \dots z_{r_D}$:

$$\begin{cases} \partial_t z_B = \partial_{xx}^2 z_B + K z_B \text{ in } Q_T, \\ z_B(t,0) = e_1 \hat{v}_1(t), \quad z_B(t,1) = -e_{\tilde{s}_1} \hat{v}_2(t) \text{ on } (0,T). \end{cases} \tag{22}$$

Assume for the moment that K satisfies the boundary Kalman condition

$$\operatorname{rank}\left[\mathcal{K}_q \,:\, \left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm}\right)\right]_{\tilde{s}_B q} = \tilde{s}_B q, \quad \forall q \in \mathbb{N}^*, \tag{23}$$

where we recall that

$$\mathcal{K}_q = \begin{bmatrix} K_1 & 0 & \cdots & \cdots & 0 \\ 0 & K_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & K_q \end{bmatrix} \in \mathcal{M}_{\tilde{s}_B q}(\mathbb{R}) \text{ and } K_k = -\lambda_k I + K \in \mathcal{M}_{\tilde{s}_B}(\mathbb{R}).$$

Then, we deduce that the system (22) is null-controllable. In particular, let a time $T_B \in (0,T)$ be given, then there exist controls $\hat{\hat{v}}_1, \hat{\hat{v}}_2 \in L^2(0,T_B)$ such that $z_B(T_B) = 0$ in Ω. We choose

$$\hat{v}_1(t) = \begin{cases} \hat{\hat{v}}_1(t) \text{ if } t \in (0,T_B), \\ 0 \text{ otherwise.} \end{cases} \quad \hat{v}_2(t) = \begin{cases} \hat{\hat{v}}_2(t) \text{ if } t \in (0,T_B), \\ 0 \text{ otherwise.} \end{cases}$$

as controls, and one can see that $z_B(t) = 0$ in Ω for all $t \geq T_B$. As a consequence $\hat{z}$ defined by

$$\hat{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_{r_D} \end{bmatrix}$$

satisfies

$$\begin{cases} \partial_t \hat{z} = \partial_{xx}^2 \hat{z} + \hat{C}\hat{z} + e_{S_1} \hat{u}_{i_1}(t,x) 1_{\omega_{i_1}}(x) + \cdots + e_{S_{r_D}} \hat{u}_{i_{r_D}}(t,x) 1_{\omega_{i_{r_D}}}(x) \text{ in } Q_{(T_B,T)}, \\ \hat{z}(t,0) = 0, \quad \hat{z}(t,1) = 0 \text{ on } (T_B,T), \end{cases}$$

with

$$\hat{C} = \begin{bmatrix} C_1 & \times & \cdots & \cdots & \times \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \times \\ 0 & \cdots & \cdots & 0 & C_{r_D} \end{bmatrix}.$$

And since for all $i \in \{1, \dots, r_D\}$ the distributed Kalman condition rank $[\, C_i \ : \ e_i \,]_{s_i} = s_i$ is satisfied we can iterate this process and this will lead the result.

As a consequence, it remains to prove that the condition (23) is satisfied:

Step 3 In fact, condition (23) holds if and only if for all $q \in \mathbb{N}^*$ the Hautus test holds (see Proposition 2):

$$\ker\left(\mathcal{K}_q^* - \theta I\right) \cap \ker\left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm}\right)^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{24}$$

To prove (24) we will use condition (8), and to this aim let us then first reformulate (24) in terms of the original data of the problem, that is A, D, B_1 and B_2. This is done through the following lemma:

Lemma 2 *Assume that condition* (8) *holds. We have the following equivalences:*

1. *For all $q \in \mathbb{N}^*$ the Hautus test* (24) *holds.*
2. *For all $q \in \mathbb{N}^*$, for all $\theta \in \mathbb{C}$, for all $s \in \mathbb{N}^*$ and for all $V^1, \dots, V^s \in \mathbb{R}^{\tilde{s}_B q}$ linearly independent vectors of* $\ker\left(\mathcal{K}_q^* - \theta I\right)$, *the set*

$$\left\{\left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm}\right)^* V^k\right\}_{1 \le k \le s}$$

   is linearly independent in $\mathbb{R}^{2q}$.
3. *For all $q \in \mathbb{N}^*$, for all $\theta \in \mathbb{C}$, for all $s \in \mathbb{N}^*$ and for all $W^1, \dots, W^s \in \mathbb{R}^{nq}$ linearly independent vectors of* $\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1} | \cdots | D_{i_{r_D}}\right)\right)_q^*$, *the set*

$$\left\{\left(\mathsf{B}_{\mathsf{j}_1} \middle| \mathsf{B}_{\mathsf{j}_2}\right)^* W^k\right\}_{1 \le k \le s}$$

   is linearly independent in $\mathbb{R}^{2q}$.

And those conditions are implied by the following one: for all $q \in \mathbb{N}^*$ we have

$$\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1} | \cdots | D_{i_{r_D}}\right)\right)_q^* \cap \ker\left(\mathsf{B}_{\mathsf{j}_1} \middle| \mathsf{B}_{\mathsf{j}_2}\right)^* = \{0\}, \quad \forall \theta \in \mathbb{C}. \tag{25}$$

Proof One can see that item 1 is equivalent to item 2 (see for instance [6, Proposition 3.1]) and that condition (25) implies item 3. Thus let us prove that item 2 and item 3 are equivalent. Let us fix $q \in \mathbb{N}^*$ and $\theta \in \mathbb{C}$. We define a bijective linear map Φ by:

$$\begin{aligned}\Phi : \ker\left(\mathcal{K}_q^* - \theta I\right) &\longrightarrow \ker\left(\mathcal{A}_q^* - \theta I\right) \cap E \\ V = \begin{bmatrix} V_1 \\ \vdots \\ V_q \end{bmatrix} &\longmapsto \begin{bmatrix} \tilde{\Phi}(V_1) \\ \vdots \\ \tilde{\Phi}(V_q) \end{bmatrix}\end{aligned}$$

where

$$\tilde{\Phi}(V_l) = (P_1^*)^{-1} \begin{bmatrix} 0 \\ V_l \end{bmatrix} \in \mathbb{R}^n,$$

and

$$E = \left\{ W = \begin{bmatrix} W_1 \\ \vdots \\ W_q \end{bmatrix} \in \mathbb{R}^{nq} \text{ such that } \forall l \in \{1, \dots, q\}, \quad W_l = (P_1^*)^{-1} \begin{bmatrix} 0 \\ \times \end{bmatrix} \right\}.$$

One can check that in fact $E = \ker(P_q^D)^*$ and thus $\ker\left(\mathcal{A}_q^* - \theta I\right) \cap E = \ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q^*$. Moreover, for all $s \in \mathbb{N}^*$ and all $\{\alpha_k\}_{1 \le k \le s} \subset \mathbb{R}$ we have

$$\begin{aligned}\sum_{k=1}^{s} \alpha_k \left(\mathsf{B}_{\mathsf{j}_1} | \mathsf{B}_{\mathsf{j}_2}\right)^* \Phi(V^k) &= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \left(B_1 \middle| (-1)^l B_2 \right)^* \Phi_l(V_l^k) \right) \\ &= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \left(e_{\tilde{S}_1} \middle| (-1)^{l+1} e_{\tilde{S}_2} \right)^* P_1^* \Phi_l(V_l^k) \right) \\ &= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \left(e_{\tilde{S}_1} \middle| (-1)^{l+1} e_{\tilde{S}_2} \right)^* \begin{bmatrix} 0 \\ V_l^k \end{bmatrix} \right) \\ &= \sum_{k=1}^{s} \alpha_k \left(\sum_{l=1}^{q} \left(e_1 \middle| (-1)^{l+1} e_{\tilde{s}_1} \right)^* V_l^k \right) \\ &= \sum_{k=1}^{s} \alpha_k \left((e_1)_q \middle| (-e_{\tilde{s}_1})_q^{\pm} \right)^* V^k.\end{aligned}$$

Combining those two facts, the claim is proved. □

As a consequence of Lemma 2, it is sufficient to prove that (25) is true, and in fact it is a consequence of the Hautus test, and hypothesis (8):

Lemma 3 *Assume that condition* (8) *holds. Then, for all* $q \in \mathbb{N}^*$ *we have*

$$\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q^* \cap \ker\left(\mathsf{B}_{\mathsf{j}_1}\middle|\mathsf{B}_{\mathsf{j}_2}\right)^* = \{0\}, \quad \forall\theta \in \mathbb{C}. \tag{26}$$

Proof (26) can be rewritten as

$$\ker\left(\mathcal{A}_q^* - \theta I\right) \cap \ker\left(\mathsf{B}_{\mathsf{j}_1}\middle|\mathsf{B}_{\mathsf{j}_2}\middle|\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q\right)^* = \{0\}, \quad \forall\theta \in \mathbb{C}.$$

which is equivalent to (by the Hautus test)

$$\operatorname{rank}\left[\mathcal{A}_q \;:\; \left(\mathsf{B}_{\mathsf{j}_1}\middle|\mathsf{B}_{\mathsf{j}_2}\middle|\left(\left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right)_q\right)\right]_{nq} = nq,$$

and this last formulation is also equivalent to

$$\operatorname{rank}\left[\left[\mathcal{A}_q \;:\; \left(\mathsf{B}_{\mathsf{j}_1}\middle|\mathsf{B}_{\mathsf{j}_2}\right)\right]_{nq}\middle|\left(\left(\left[A \;:\; \left(D_{i_1}|\cdots|D_{i_{r_D}}\right)\right]_n\right)\right)_q\right] = nq, \tag{27}$$

in the same way as condition (8) is equivalent to condition (9) (see Remark 1, item 2). Now thanks to Lemma 1 we can see that (27) holds (observe that we have more powers of $\mathcal{A}_q$ and A in (27)). □

Let us now prove the necessary part of Theorem 1:

Step 4 Suppose that there exists $q_0 \in \mathbb{N}^*$ such that

$$\operatorname{rank}\left[\left[\mathcal{A}_{q_0} \;:\; \left((B_1)_{q_0}\middle|(B_2)_{q_0}^{\pm}\right)\right]_{nq_0}\middle|\left(([A \;:\; D]_n)\right)_{q_0}\right] < nq_0.$$

Thanks to the other characterization of condition (8) (see Remark 1, item 2) this means we have

$$\operatorname{rank}\left[\mathcal{A}_{q_0} \;:\; \left((B_1)_{q_0}\middle|(B_2)_{q_0}^{\pm}\middle|((D))_{q_0}\right)\right]_{nq_0} < nq_0. \tag{28}$$

Thus, there exists $\Psi^T \in \mathbb{R}^{nq_0}$ with $\Psi^T \neq 0$ such that $\Psi(t) = e^{\mathcal{A}_{q_0}^*(T-t)}\Psi^T$ satisfies (see Proposition 2)

$$\left((B_1)_{q_0}\middle|(B_2)_{q_0}^{\pm}\middle|((D))_{q_0}\right)^*\Psi(t) = 0, \quad \forall t \in [0, T]. \tag{29}$$

Let us write $\Psi(t)$ as follow:

$$\Psi(t) = e^{\mathcal{A}^*_{q_0}(T-t)}\Psi^T = \begin{bmatrix} e^{A_1^*(T-t)} & 0 & \cdots & \cdots & 0 \\ 0 & e^{A_2^*(T-t)} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & e^{A_{q_0}^*(T-t)} \end{bmatrix} \begin{bmatrix} \Psi_1^T \\ \Psi_2^T \\ \vdots \\ \vdots \\ \Psi_{q_0}^T \end{bmatrix}$$

$$= \begin{bmatrix} e^{(-\lambda_1 I + A^*)(T-t)}\Psi_1^T \\ e^{(-\lambda_2 I + A^*)(T-t)}\Psi_2^T \\ \vdots \\ \vdots \\ e^{(-\lambda_{q_0} I + A^*)(T-t)}\Psi_{q_0}^T \end{bmatrix} = \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \vdots \\ \vdots \\ \Psi_{q_0}(t) \end{bmatrix}, \quad \forall t \in [0, T].$$

Thus (29) gives

$$\begin{bmatrix} B_1^* & B_1^* & \cdots & \cdots & \cdots & B_1^* \\ -B_2^* & B_2^* & \cdots & \cdots & \cdots & (-1)^{q_0} B_2^* \\ D^* & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & D^* & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ 0 & \cdots & \cdots & & 0 & D^* \end{bmatrix} \begin{bmatrix} \Psi_1(t) \\ \Psi_2(t) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \Psi_{q_0}(t) \end{bmatrix} = 0, \quad \forall t \in [0, T],$$

i.e.

$$\forall t \in [0, T], \quad \begin{cases} \displaystyle\sum_{k=1}^{q_0} B_1^* \Psi_k(t) = 0, \\ \displaystyle\sum_{k=1}^{q_0} (-1)^k B_2^* \Psi_k(t) = 0, \\ D^* \Psi_k(t) = 0, \quad \forall k \in \{1, \ldots, q_0\}. \end{cases} \tag{30}$$

Let us now prove that the observability inequality (19) fails. To this aim we define $\Phi^T \in H_0^1(\Omega; \mathbb{R}^n)$ by

$$\Phi^T = \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} \Psi_k^T \phi_k.$$

and let Φ be the solution to

$$\begin{cases} -\partial_t \Phi = \partial_{xx}^2 \Phi + A^* \Phi \text{ in } Q_T, \\ \Phi = 0 \text{ on } \Sigma_T, \\ \Phi(T) = \Phi^T \text{ in } \Omega, \end{cases}$$

i.e.

$$\begin{aligned} \Phi(t) &= \sum_{k=1}^{q_0} e^{(-\lambda_k I + A^*)(T-t)} \begin{bmatrix} \langle \Phi_1^T, \phi_k \rangle_{L^2} \\ \vdots \\ \langle \Phi_n^T, \phi_k \rangle_{L^2} \end{bmatrix} \phi_k \\ &= \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} e^{(-\lambda_k I + A^*)(T-t)} \Psi_k^T \phi_k \\ &= \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} \Psi_k(t) \phi_k. \end{aligned} \tag{31}$$

From (30) we obtain

$$\forall t \in (0, T), \forall i \in \{1, \dots, n_D\}, \quad D_i^* \Phi(t) = \sum_{k=1}^{q_0} \lambda_k{}^{-1/2} D_i^* \Psi_k(t) \phi_k = 0 \text{ in } \Omega,$$

and since the space dimension is $N = 1$ we also obtain

$$\forall t \in (0, T), \quad B_1^* \partial_x \Phi(t, 0) = B_1^* \sum_{k=1}^{q_0} \Psi_k(t) \underbrace{\lambda_k{}^{-1/2} \partial_x \phi_k(0)}_{=1} = 0,$$

and

$$\forall t \in (0, T), \quad B_2^* \partial_x \Phi(t, 1) = B_2^* \sum_{k=1}^{q_0} \Psi_k(t) \underbrace{\lambda_k{}^{-1/2} \partial_x \phi_k(1)}_{=(-1)^k} = 0.$$

Finally, let us remark that $\Phi(0) \neq 0$ since $\Psi^T \neq 0$ (see (31)). As a consequence, the observability inequality (19) fails and so does the null-controllability of (16). □

4 The Carleman estimate

Recall that all along this section, no boundary controls are considered and the space dimension N is arbitrary. The aim is to prove Theorem 2 and as said before this latter is a consequence of a Carleman estimate.

4.1 A Carleman estimate for cascade systems

Let us introduce the framework in which the Carleman estimate will be established. We consider this time the following $n \times n$ parabolic system:

$$\begin{cases} \partial_t y = \Delta y + Cy + e_{S_1} u_1(t,x) 1_{\omega_1}(x) + \cdots + e_{S_r} u_r(t,x) 1_{\omega_r}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases} \tag{32}$$

where $r \in \{1, \ldots, n\}$ is the number of controls, $u_1 \in L^2(Q_T), \ldots, u_r \in L^2(Q_T)$ are the controls and where $C = C(t,x) \in L^\infty(Q_T; \mathcal{M}_n(\mathbb{R}))$ is a matrix with the following block cascade type structure:

$$C = \begin{bmatrix} C_{11} & \times & \times & \cdots & \times \\ 0 & C_{22} & \times & \cdots & \times \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \times \\ 0 & \cdots & \cdots & 0 & C_{rr} \end{bmatrix} \text{ with } C_{jj} = \begin{bmatrix} c^j_{11} & c^j_{12} & c^j_{13} & \cdots & c^j_{1s_j} \\ c^j_{21} & c^j_{22} & c^j_{23} & \cdots & c^j_{2s_j} \\ 0 & c^j_{32} & c^j_{33} & \cdots & c^j_{3s_j} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c^j_{s_j s_{j-1}} & c^j_{s_j s_j} \end{bmatrix}, \tag{33}$$

where $s_j \in \mathbb{N}$ is the size of the bloc C_{jj} (in particular we have $\sum_{j=1}^r s_j = n$) and where $S_1, \ldots, S_{r+1}$ are such that one control is exerted on each first equation of a block, this reads $S_i = 1 + \sum_{j=1}^{i-1} s_j$ for all $i \in \{1, \ldots, r+1\}$.

Those notations in mind, we have

Theorem 3 *Assume that for every $j \in \{1, \ldots, r\}$ there exists a non-empty open subset $\tilde{\omega}_j \subset \omega_j$ and $\underline{c}^j > 0$ such that for every $i \in \{1, \ldots, s_j\}$ we have*

$$c^j_{i+1,i} \geq \underline{c}^j \text{ or } -c^j_{i+1,i} \geq \underline{c}^j \text{ on } (0,T) \times \tilde{\omega}_j. \tag{34}$$

Then, there exist functions $\beta_1, \ldots, \beta_r \in C^2(\overline{\Omega})$ such that $0 < \beta_1 < \cdots < \beta_r$, there exists $\mathbf{C} > 0$, $\mathbf{s_0} > 0$ and $\mathbf{l_0} > 0$ such that, for every $\Phi^T \in L^2(\Omega; \mathbb{R}^n)$, the solution Φ to the system

$$\begin{cases} -\partial_t \Phi = \Delta\Phi + C^*\Phi \text{ in } Q_T, \\ \Phi = 0 \text{ on } \Sigma_T, \\ \Phi(T) = \Phi^T \text{ in } \Omega, \end{cases} \tag{35}$$

satisfies

$$\sum_{j=1}^r \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \mathbf{C} \sum_{j=1}^r \iint_{(0,T)\times\omega_j} (s\varphi)^{\mathbf{l_0}} e^{-2s\eta_j} |\Phi_{S_j}|^2,$$

for all $s \geq \mathbf{s_0}$. *Here we have denoted*

$$\mathcal{J}_j(d,q) = \iint\limits_{Q_T} (s\varphi)^{d-2} e^{-2s\eta_j} |\nabla q|^2 + \iint\limits_{Q_T} (s\varphi)^d e^{-2s\eta_j} |q|^2, \tag{36}$$

and $\varphi(t) = (t(T-t))^{-1}$, $\eta_j(t,x) = \beta_j(x)\varphi(t)$ *for* $j \in \{1,\dots,r\}$.

Proof We adapt the proof of [10, Theorem 1.1], but we consider different weight functions on each block and we use the fact that we still can choose such functions in an ordered way. First, let us rewrite system (35) on each block as follows:

$$\forall j \in \{1,\dots,r\}, \quad \begin{cases} -\partial_t \Phi_i = \Delta\Phi_i + \sum\limits_{k=1}^{i} c_{ki}\Phi_k + c_{i+1,i}\Phi_{i+1} \text{ in } Q_T, \\ \qquad \forall i \in \{S_j,\dots S_{j+1}-2\} \\ -\partial_t \Phi_{S_{j+1}-1} = \Delta\Phi_{S_{j+1}-1} + \sum\limits_{k=1}^{S_{j+1}-1} c_{k,S_{j+1}-1}\Phi_k \text{ in } Q_T, \\ \Phi_i = 0 \text{ on } \Sigma_T, \quad \forall i \in \{S_j,\dots,S_{j+1}-1\}. \end{cases} \tag{37}$$

where we rewrote for convenience $C = (c_{ij})_{1\leq i,j\leq n}$. And let us recall the following Carleman estimate for one single parabolic equation (see [12, Lemma 2.3]):

Lemma 4 *Let* $\omega \subset \Omega$ *be a non-empty open subset. For every* $\underline{\beta} > 0$, *there exists a function* $\beta \in C^2(\overline{\Omega})$ *such that* $\beta > \underline{\beta}$ *and such that, for every* $d \in \mathbb{R}$, *there exist* $\mathbf{C} > 0$, $\mathbf{s_0} > 0$ *such that, for every* $\psi^T \in L^2(\Omega)$ *and every* $f \in L^2(Q_T)$, *the solution* ψ *to*

$$\begin{cases} -\partial_t\psi = \Delta\psi + f(t,x) \text{ in } Q_T, \\ \psi = 0 \text{ on } \Sigma_T, \\ \psi(T) = \psi^T \text{ in } \Omega, \end{cases}$$

satisfies

$$\iint\limits_{Q_T} (s\varphi)^{d-2} e^{-2s\eta} |\nabla\psi|^2 + \iint\limits_{Q_T} (s\varphi)^d e^{-2s\eta} |\psi|^2$$
$$\leq \mathbf{C}\left(\iint\limits_{(0,T)\times\omega} (s\varphi)^d e^{-2s\eta} |\psi|^2 + \iint\limits_{Q_T} (s\varphi)^{d-3} e^{-2s\eta} |f|^2 \right)$$

for all $s \geq \mathbf{s_0}$. *Where* $\eta(t,x) = \beta(x)\varphi(t)$.

To start the proof of Theorem 3, let be given $\tilde{\tilde{\omega}}_1 \subset\subset \tilde{\omega}_1, \ldots, \tilde{\tilde{\omega}}_p \subset\subset \tilde{\omega}_p$. For every $j \in \{1, \ldots, r\}$ we apply Lemma 4 with $\omega = \tilde{\tilde{\omega}}_j$ which enables us to construct functions $\beta_1, \ldots, \beta_r \in C^2(\overline{\Omega})$ such that $0 < \beta_1 < \cdots < \beta_r$ and such that each function Φ_i, $S_j \leq i \leq S_{j+1} - 1$, satisfies, due to the particular structure of C^* (see (37)):

$$\forall i \neq S_{j+1} - 1, \quad \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \frac{1}{2}\mathbf{C_1}\Bigg(\mathcal{L}_j\left(\tilde{\tilde{\omega}}_j; 3(S_{j+1} - i), \Phi_i\right) \\ + \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_{i+1})) + \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k)\Bigg),$$

and

$$\mathcal{J}_j(3, \Phi_{S_{j+1}-1}) \leq \frac{1}{2}\mathbf{C_1}\left(\mathcal{L}_j\left(\tilde{\tilde{\omega}}_j; 3(S_{j+1} - i), \Phi_{S_{j+1}-1}\right) + \sum_{k=1}^{S_{j+1}-1} \mathcal{J}_j(0, \Phi_k)\right),$$

for s large enough, where here and in what follows we denote

$$\mathcal{L}_k(\omega; d, q) = \iint_{(0,T)\times\omega} (s\varphi)^d e^{-2s\eta_k} |q|^2.$$

Summing over i this gives

$$\sum_{i=S_j}^{S_{j+1}-1} \mathbf{C_1}^{i-S_j} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \sum_{i=S_j}^{S_{j+1}-1} \frac{1}{2}\mathbf{C_1}^{i+1-S_j}\Bigg(\mathcal{L}_j\left(\tilde{\tilde{\omega}}_j; 3(S_{j+1} - i), \Phi_i\right) \\ + \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k)\Bigg).$$

Summing over j this leads to

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \mathbf{C_2}\Bigg(\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{L}_j\left(\tilde{\tilde{\omega}}_j; 3(S_{j+1} - i), \Phi_i\right) \\ + \sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \sum_{k=1}^{i} \mathcal{J}_j(3(S_{j+1} - 1 - i), \Phi_k)\Bigg). \tag{38}$$

Now let us remark that the last term in the right hand-side of (38) can be seen as the sum of two terms:

$$\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\sum_{k=1}^{i}\mathcal{J}_j(3(S_{j+1}-1-i),\Phi_k)=\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\mathcal{J}_j(3(S_{j+1}-1-i),\Phi_i)$$
$$+\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\sum_{k=1}^{i-1}\mathcal{J}_j(3(S_{j+1}-1-i),\Phi_k),$$

and for s large enough the first one can be absorbed by the left hand-side of (38), because $S_{j+1}-1-i<S_{j+1}-i$, and the second one can also be absorbed by the left hand-side of (38) since the functions β_j have been constructed such that

$$0<\beta_1<\beta_2<\cdots<\beta_r,$$

so that, for every $d_1,d_2\geq 0$, there exists $\mathbf{C}>0$ such that

$$(s\varphi)^{d_1}e^{-2s\eta_j}\leq \mathbf{C}(s\varphi)^{d_2}e^{-2s\eta_{j-1}},\quad \forall j\in\{2,\ldots,r\}$$

for s large enough. Thus, we finally obtain

$$\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\mathcal{J}_j(3(S_{j+1}-i),\Phi_i)\leq \mathbf{C_3}\left(\sum_{j=1}^{r}\sum_{i=S_j}^{S_{j+1}-1}\mathcal{L}_j\left(\tilde{\tilde{\omega}}_j;3(S_{j+1}-i),\Phi_i\right)\right)\tag{39}$$

for s large enough. To pursue we use the following lemma (see [10, Section 4]):

Lemma 5 *Assume that* (34) *holds. Then, for every* $\epsilon>0$, $j\in\{1,\ldots r\}$, $i\in\{S_j+1,\ldots,S_{j+1}-1\}$ *and* $l\in\mathbb{N}$ *and every open sets* $\mathcal{O}_0,\mathcal{O}_1$ *such that* $\tilde{\tilde{\omega}}_j\subset\mathcal{O}_1\subset\subset\mathcal{O}_0\subset\tilde{\omega}_j$, *there exist* $\mathbf{C}>0$, $\mathbf{s_0}>0$ *and* $l_1(j,l),\ldots,l_{i-1}(j,l)\in\mathbb{N}$ *such that the solution* Φ *to (35) satisfies*

$$\forall i\neq S_{j+1}-1,\quad \mathcal{L}_j(\mathcal{O}_1;l,\Phi_i)\leq\epsilon\left(\mathcal{J}_j(3(S_{j+1}-i),\Phi_i)+\mathcal{J}_j(3(S_{j+1}-1-i),\Phi_{i+1})\right)+\mathbf{C}\sum_{k=1}^{i-1}\mathcal{L}_j(\mathcal{O}_0;l_k(j,l),\Phi_k).$$

and

$$\mathcal{L}_j(\mathcal{O}_1;l,\Phi_{S_{j+1}-1})\leq\epsilon\mathcal{J}_j(3,\Phi_{S_{j+1}-1})+\mathbf{C}\sum_{k=1}^{S_{j+1}-2}\mathcal{L}_j(\mathcal{O}_0;l_k(j,l),\Phi_k).$$

for all $s\geq\mathbf{s_0}$.

For each $j \in \{1, \ldots, r\}$ let be given $\tilde{\tilde{\omega}}_j \subset\subset \mathcal{O}_{j,1} \subset\subset \cdots \subset\subset \mathcal{O}_{j,s_j-1} \subset\subset \tilde{\omega}_j$. Applying Lemma 5 to $i = S_{j+1} - 1$, $\mathcal{O}_1 = \tilde{\tilde{\omega}}_j$, $\mathcal{O}_0 = \mathcal{O}_{j,1}$, $l = 3(S_{j+1} - i) = l_j^1$ and $\epsilon = \frac{1}{2\mathbf{C_3}}$, we have

$$\mathcal{L}_j(\tilde{\tilde{\omega}}_j; l_j^1, \Phi_{S_{j+1}-1}) \leq \frac{1}{2\mathbf{C_3}} \mathcal{J}_j(3, \Phi_{S_{j+1}-1}) + \mathbf{C_4} \sum_{k=1}^{S_{j+1}-2} \mathcal{L}_j(\mathcal{O}_{j,1}; l_k(j, l_j^1), \Phi_k).$$

Back to (39) we obtain

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \mathbf{C_5} \left(\sum_{j=1}^{r} \sum_{k=1}^{S_{j+1}-2} \mathcal{L}_j(\mathcal{O}_{j,1}; \max\left\{3(S_{j+1} - k), l_k(j, l_j^1)\right\}, \Phi_k) \right).$$

Applying now Lemma 5 to $i = S_{j+1} - 2$, $\mathcal{O}_1 = \mathcal{O}_{j,1}$, $\mathcal{O}_0 = \mathcal{O}_{j,2}$, $l = \max\left\{6, l_{S_{j+1}-2}(j, l_j^1)\right\} = l_j^2$ and $\epsilon = \frac{1}{2\mathbf{C_5}}$, we obtain

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \mathbf{C_6} \left(\sum_{j=1}^{r} \sum_{k=1}^{S_{j+1}-3} \mathcal{L}_j(\mathcal{O}_{j,2}; \max\left\{\max\left\{3(S_{j+1} - k), l_k(j, l_j^1)\right\}, l_k(j, l_j^2)\right\}, \Phi_k) \right).$$

Iterating the process leads to the estimate

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \mathbf{C_7} \left(\sum_{j=1}^{r} \sum_{k=1}^{S_j} \mathcal{L}_j(\mathcal{O}_{j,s_j-1}; \max\left\{3(S_{j+1}-k), l_k(j, l_j^1), \ldots, l_k(j, l_j^{s_j-1})\right\}, \Phi_k) \right),$$

which can be rewritten as follow by separating the term $k = S_j$

$$\sum_{j=1}^{r} \sum_{i=S_j}^{S_{j+1}-1} \mathcal{J}_j(3(S_{j+1} - i), \Phi_i) \leq \mathbf{C_7} \left(\sum_{j=1}^{r} \mathcal{L}_j(\mathcal{O}_{j,s_j-1}; \max\left\{3s_j, l_{S_j}(j, l_j^1), \ldots, l_{S_j}(j, l_j^{s_j-1})\right\}, \Phi_{S_j}) \right.$$

$$+\sum_{j=1}^{r}\sum_{k=1}^{S_j-1}\mathcal{L}_j(\mathcal{O}_{j,s_j-1};\max\left\{3(S_{j+1}-k),l_k(j,l_j^1),\ldots,l_k(j,l_j^{s_j-1})\right\},\Phi_k)\Bigg). \tag{40}$$

Let us now recall that we have chosen $\beta_1 < \beta_2 < \cdots < \beta_r$ so that the last term in the right-hand side of (40) can be absorbed, for s large enough, by the term of the left-hand side of (40), no matter what the power of s in those terms are. □

4.2 Proof of Theorem 2

All the work is based on to the previous Carleman estimate. Indeed, following the ideas of [4] we can construct a change of basis thanks to condition (15) which leads to a cascade system (see [4, Lemma 4.1]) with possibly controls acting on different subdomains. Applying the previous Carleman estimate we deduce the result.

5 Further results and comments

1. Until now, we looked at the null-controllability properties for systems where the coefficients in front of the operator $-\Delta$ were the same on every equation but we can also consider systems where those coefficients are different; let us consider

$$\begin{cases} \partial_t y = J\Delta y + Ay + D_1 u_1(t,x)1_{\omega_1}(x) + \cdots + D_{n_D}u_{n_D}(t,x)1_{\omega_{n_D}}(x) \text{ in } Q_T, \\ y = 0 \text{ on } \Sigma_T, \end{cases} \tag{41}$$

with $J \in \mathcal{M}_n(\mathbb{R})$ such that J is diagonalizable with positive eigenvalues. Then there exists also a Kalman rank condition, which is a simple extension of the one proved in [2]:

Theorem 4 *Under the previous assumption on J, system (41) is null-controllable if and only if*

$$\text{rank}\,[\,-\lambda_k J + A\ :\ D\,]_n = n, \quad \forall k \in \mathbb{N}^*. \tag{42}$$

When J is the identity matrix condition (42) is indeed equivalent to condition (13) since $\text{rank}\,[\,-\lambda_k I + A\ :\ D\,]_n = \text{rank}\,[\,A\ :\ D\,]_n$ for all $k \in \mathbb{N}^*$.
Theorem 4 can be proved by slightly changing the proof of Theorem 1.1 of [2]. Indeed, with the notations of [2], changing the definition of $\mathcal{K}$ one can see that Theorem 1.1 is still a consequence of Theorem 1.3 and Theorem 1.3 is still a consequence of Theorem 1.2 and Theorem 2.1. The proof of Theorem 2.1 remains unchanged and the proof of Theorem 1.2 can be adapted to our case by applying Theorem 3.2 to $\phi = D_i^*\varphi$ instead of $\phi = (B^*\varphi)_i$.

2. All the results of Sect. 2.2 and Theorem 4 remain true if we replace the operator $-\Delta$ by more general elliptic operators $-R$, of the form

$$\begin{cases} Ry = \sum_{i,j=1}^{N} \partial_i \left(r_{ij}(x) \partial_j y \right), \\ r_{ij} \in W^{1,\infty}(\Omega), \quad r_{ij} = r_{ji} \text{ in } \Omega, \quad \forall i, j \in \{1, \dots, N\}, \\ \exists\, \underline{r} > 0, \quad \sum_{i,j=1}^{N} r_{ij} \xi_i \xi_j \geq \underline{r} |\xi|^2 \text{ in } \Omega, \quad \forall \xi \in \mathbb{R}^N. \end{cases}$$

3. In this paper, we used a particular strategy which does not apply to many other problems. For instance, the case of space varying coefficients cannot be analyzed the same way. Indeed we did a change of variable, so we used the fact that Δ commutes with P_1.
   As our result is based on the one of [6], the N-dimensional case with $N > 1$ is still open.

Acknowledgments I would like to thank Assia Benabdallah and Franck Boyer for their advices and remarks.

References

1. Alabau-Boussouira F, Léautaud M (2011) Indirect controllability of locally coupled systems under geometric conditions. C R Acad Sci Paris 349(7–8):395–400
2. Ammar-Khodja F, Benabdallah A, Dupaix C (2006) Null-controllability of some reaction-diffusion systems with one control force. J Math Anal Appl 320(2):928–943
3. Ammar-Khodja F, Benabdallah A, Dupaix C, González-Burgos M (2009) A Kalman rank condition for the localized distributed controllability of a class of linear parbolic systems. J Evol Equ 1(2):267–291
4. Ammar-Khodja F, Benabdallah A, Dupaix C, González-Burgos M (2009) A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems. Differ Equ Appl 1(3):139–151
5. Ammar-Khodja F, Benabdallah A, Dupaix C, Kostine I (2005) Null controllability of some systems of parabolic type by one control force. ESAIM Control Optim Calc Var 11(3):426–448
6. Ammar-Khodja F, Benabdallah A, González-Burgos M, de Teresa L (2011) The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials. Journal de Mathématiques Pures et Appliquées 96(6):555–590. http://www.sciencedirect.com/science/article/pii/S002178241100078X
7. de Teresa L (2000) Insensitizing controls for a semilinear heat equation. Commun Partial Differ Equ 25(1–2):39–72
8. Fernández-Cara E, González-Burgos M, de Teresa L (2010) Boundary controllability of parabolic coupled equations. J Funct Anal 259(7):1720–1758
9. González-Burgos M, Pérez-García R (2006) Controllability results for some nonlinear coupled parabolic systems by one control force. Asymptot Anal 46:123–162
10. González-Burgos M, de Teresa L (2010) Controllability results for cascade systems of *m* coupled parabolic PDEs by one control force. Port Math 67(1):91–113
11. Guerrero S (2007) Null controllability of some systems of two parabolic equations with one control force. SIAM J Control Optim 46(2):379–394
12. Imanuvilov OY, Yamamoto M (2003) Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations. Publ Res Inst Math Sci 39(2):227–274
13. Tucsnak M, Weiss G (2009) Observation and control for operator semigroups, Advanced Texts, Birkhäuser, Basel