EVOLUTION EQUATIONS AND
CONTROL THEORY
Volume 3, Number 1, March 2014

doi:10.3934/eect.2014.3.167

pp. 167–189

BOUNDARY APPROXIMATE CONTROLLABILITY OF SOME LINEAR PARABOLIC SYSTEMS

Guillaume Olive

LATP, UMR 7353, Aix-Marseille université
Technopôle Château-Gombert, 39, rue F. Joliot-Curie
13453 Marseille cedex 13, France

(Communicated by Thomas Seidman)

Abstract. This paper focuses on the boundary approximate controllability of two classes of linear parabolic systems, namely a system of n heat equations coupled through constant terms and a 2×2 cascade system coupled by means of a first order partial differential operator with space-dependent coefficients.

For each system we prove a sufficient condition in any space dimension and we show that this condition turns out to be also necessary in one dimension with only one control. For the system of coupled heat equations we also study the problem on rectangle, and we give characterizations depending on the position of the control domain. Finally, we prove the distributed approximate controllability in any space dimension of a cascade system coupled by a constant first order term.

The method relies on a general characterization due to H.O. Fattorini.

1. **Introduction.** The controllability of parabolic systems is a difficult problem. While Carleman estimates have been successfully used to prove the distributed null-controllability of some linear parabolic systems (e.g. [2, 15, 16, 22, 8]), there are still many cases where these estimates appear to be of no help. An example of such situation is when the control domain and the coupling domain do not meet each other ([1, 25, 9]). The boundary controllability is another of these situations and requires new techniques to be solved. In [14] and [4], the authors developped the method of moments of H.O. Fattorini and D.L. Russell to establish a characterization of the boundary null-controllability in dimension 1 for a system of n coupled heat equations. In [1], the authors used transmutation techniques to obtain a boundary null-controllability result in any dimension for a system of 2 heat equations, with a particular coupling. Finally, in [19] the authors proved the boundary approximate controllability of a cascade system of 2 heat equations in any dimension by developping the solution into Fourier series. To the author knowledge, these results are the only ones concerning the boundary controllability of linear parabolic systems of heat-type. For more details, a good account on actual methods and recent open problems for the distributed or boundary controllability of linear parabolic systems we refer to the survey [5].

2010 *Mathematics Subject Classification.* 93B05, 93C05, 35K05.
Key words and phrases. Parabolic systems, boundary controllability, distributed controllability, Hautus test.

In the present work we are interested in the boundary approximate controllability of two classes of linear parabolic systems introduced in [14] and [19]. More precisely, the first system we study is the following [1]

$$\begin{cases} \partial_t y - \Delta y = Ay & \text{in } (0,T)\times\Omega, \\ y = 1_\gamma B g & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $T > 0$, Ω is a bounded open subset of $\mathbb{R}^N$, assumed regular enough, y is the state, y_0 is the initial data, A and B are $n\times n$ and $n\times m$ constant matrices with complex coefficients, g is the control, to be searched in $L^2(0,T;L^2(\partial\Omega)^m)$ - so that in fact we have m controls - and $\gamma\subset\partial\Omega$ is the control domain.

First of all, let us recall some basic facts about this kind of systems and their controllability properties:

1. System (1) is well-posed in the following sense: for every $y_0\in H^{-1}(\Omega)^n$ and $g\in L^2(0,T;L^2(\partial\Omega)^m)$, there exists a unique solution defined by transposition $y\in C^0([0,T];H^{-1}(\Omega)^n)\cap L^2(0,T;L^2(\Omega)^n)$ that depends continuously on the initial data y_0 and the control g.
2. System (1) is said to be approximately controllable at time T if for every $y_0, y_1\in H^{-1}(\Omega)^n$ and every $\epsilon > 0$, there exists a control $g\in L^2(0,T;L^2(\partial\Omega)^m)$ such that the corresponding solution y satisfies
$$||y(T)-y_1||_{H^{-1}(\Omega)^n}\leq\epsilon.$$
We say that system (1) is approximately controllable if it is approximately controllable at time T for every $T > 0$.
3. It is nowadays well-known that the controllability has a dual concept called observability and that they are linked by the following result: system (1) is approximately controllable at time T if and only if its adjoint system [2]
$$\begin{cases} \partial_t z - \Delta z = A^* z & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0) = z_0 & \text{in } \Omega, \end{cases}$$
is approximately observable at time T, that is it verifies the following unique continuation property
$$\forall z_0\in H^1_0(\Omega)^n, \quad \left(B^* 1_\gamma \partial_n z(t) = 0 \text{ for a.e. } t\in(0,T)\right) \Longrightarrow z_0 = 0. \tag{2}$$

The boundary controllability problem for system (1) has been introduced in [14]. In this paper, the authors proved a necessary and sufficient condition for this system to be null-controllable, and the same condition also characterizes the approximate controllability, see [14, Theorem 1.1] and [14, Theorem 5.2]. We point out that this work has been done in the dimension one and for 2 equations. A generalization to the case of n equations can be found in [4], still in the one-dimensional case.

[1] We will abuse the notation Δy when $y = (y_1,\dots,y_n)$ to denote $\Delta y = (\Delta y_1,\dots,\Delta y_n)$.

[2] Since the data are more regular and the system is autonomous, the solution can be taken in the sense of semigroups: $z(t) = \mathcal{S}(t)z_0$, where $\mathcal{S}(t)$ is the semigroup generated on $L^2(\Omega)^n$ by the operator $\Delta + A^*$ with domain $H^2(\Omega)^n\cap H^1_0(\Omega)^n$. Let us recall that, for $z_0\in H^1_0(\Omega)^n$, we have $z\in C^0([0,T];H^1_0(\Omega)^n)\cap L^2(0,T;H^2(\Omega)^n\cap H^1_0(\Omega)^n)$.

To the author knowledge, the only result that can be applied to system (1) in any dimension is [1, Corollary 2.2], but the matrix A has to have a very particular structure and it requires a geometric condition on γ.

In this paper we will provide conditions for the approximate controllability of this system in several interesting particular cases, see the sections 2.1 to 2.5 below. Some results are already known but we give new and simpler proofs.

The second system we deal with is the following

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t y_2 - \Delta y_2 = G(x)\cdot\nabla y_1 + a(x) y_1 & \text{in } (0,T)\times\Omega, \\ y_1 = 1_\gamma g, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega, \end{cases} \tag{3}$$

where $G \in W^{1,\infty}(\Omega)^N$, $a \in L^\infty(\Omega)$, and g is still the control, but this time we only have one control: $g \in L^2(0,T;L^2(\partial\Omega))$.

The interest in the controllability of such systems started with [19]. In this paper the authors gave sufficient conditions for the approximate controllability.

In the present work we bring a new point of view to treat this problem. This allows us to recover the result of [19] and also to provide a necessary and sufficient condition in the one-dimensional case.

The main tool to achieve our goals will be the use of a theorem of H.O. Fattorini. In fact, in 1966, H.O. Fattorini gave an interesting characterization of the approximate controllability under a general abstract framework. In his paper [12] he proved that, under some reasonable assumptions, the only observation of the eigenfunctions completely characterizes the approximate controllability. Actually, this theorem has been proved for bounded observation operators but it can easily be generalized to the case of relatively bounded observation operators as follows:

Theorem 1.1. *Let H and U be some complex Hilbert spaces. Assume that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \longrightarrow H$ generates a strongly continuous semigroup $\mathcal{S}(t)$ on H, has a compact resolvent, and the system of root vectors of its adjoint $\mathcal{A}^*$ is complete in H. Let $\mathcal{C} : \mathcal{D}(\mathcal{C}) \subset H \longrightarrow U$ be relatively bounded with respect to $\mathcal{A}$. Then, we have the property*

$$\forall z_0 \in \mathcal{D}(\mathcal{A}), \quad \Big(\mathcal{C}\mathcal{S}(t) z_0 = 0 \textit{ for a.e. } t \in (0,+\infty)\Big) \Longrightarrow z_0 = 0, \tag{4}$$

if and only if

$$\operatorname{Ker}(s-\mathcal{A}) \cap \operatorname{Ker}(\mathcal{C}) = \{0\}, \quad \forall s \in \mathbb{C}.$$

We give in appendix a proof of this theorem (which slightly changes from the one of [12]), see also [6].

Remark 1.2. Note that the condition $\operatorname{Ker}(s-\mathcal{A}) \cap \operatorname{Ker}(\mathcal{C}) = \{0\}$ can also be formulated as

$$\forall z_0 \in \operatorname{Ker}(s-\mathcal{A}), \quad \Big(\mathcal{C}\mathcal{S}(t) z_0 = 0 \text{ for a.e. } t \in (0,+\infty)\Big) \Longrightarrow z_0 = 0.$$

We use the first formulation because it is more eloquent, see Remark 1.4 below, but the most important is that Theorem 1.1 states that, in order to verify the property (4) it is enough to do so only on the eigenspaces of $\mathcal{A}$.

Remark 1.3. We will see that the operators $\mathcal{A}$ we consider generate an analytic semigroup. For instance, for the first system (1) we shall apply Theorem 1.1 to $\mathcal{A} = \Delta + A^*$. It follows from this property that $z(\cdot) = \mathcal{S}(\cdot)z_0$ is analytic in time and has a regularizing effect ($\mathcal{S}(t)z_0 \in \mathcal{D}(\mathcal{A}^\infty)$ as soon as $t > 0$, even for $z_0 \in H$). This allows us to replace in (4) the interval $(0, +\infty)$ by any interval $(0, T)$, $T > 0$, and to take the data z_0 in any space that at least contains $\mathcal{D}(\mathcal{A}^\infty)$. This shows that (2) and (4) are equivalent properties. In partictular, we see that the approximate controllability of our systems is independent of the time of control T.

Remark 1.4. When $H = \mathbb{C}^n$ and $U = \mathbb{C}^m$, $\mathcal{A} = A^*$ and $\mathcal{C} = B^*$ (where A and B are still contant matrices) this theorem can be used to prove that the ordinary differential system

$$\begin{cases} \dfrac{d}{dt}y & = & Ay + Bg \quad \text{in } (0,T), \\ y(0) & = & y_0, \end{cases}$$

is controllable[3] if and only if

$$\operatorname{Ker}(s - A^*) \cap \operatorname{Ker}(B^*) = \{0\}, \quad \forall s \in \mathbb{C}.$$

This characterization is nowadays known as the Hautus test (despite it has been proved earlier by H.O. Fattorini). M.L.J Hautus gave a direct proof of the equivalence with another characterization, the well-known Kalman rank condition (see [17, Theorem 1', §2])

$$\operatorname{rank}(B|AB|A^2B|\cdots|A^{n-1}B) = n.$$

Finally, let us mention the recent work [6] where the authors also extended the theorem of [12] in view of the stabilizability of some other parabolic systems.

Notation. We gather here some notations we shall frequently use in the sequel. We denote by $\{-\lambda_l\}_l$ the distinct Dirichlet eigenvalues of Δ on Ω. For each l, we denote by $\{\phi_{l,m}\}_m$ an orthonormal basis in $L^2(\Omega)$ of the eigenspace of Δ associated with the eigenvalue $-\lambda_l$, and by m_l the dimension of this eigenspace. It can be verified that all the following results are independent of the choice of the basis $\{\phi_{l,m}\}_m$.

All along section 2 we denote by $\{\theta_i\}_i \subset \mathbb{C}$ the distinct eigenvalues of the matrix A^* and, for each i, by $\{w_{i,j}\}_j \subset \mathbb{C}^n$ a basis of $\operatorname{Ker}(\theta_i - A^*)$. In view of section 2.3, we also denote by m_i the dimension of $\operatorname{Ker}(\theta_i - A^*)$ and we define $P_i = \left(w_{i,1}|\cdots|w_{i,m_i}\right)$. Again, one can check that all the following results do not depend on the choice of the basis $\{w_{i,j}\}_j$.

In section 3, we use the notation $\mathcal{P}_{\lambda_l}$ for the orthogonal projection in $L^2(\Omega)$ on the eigenspace of Δ associated with $-\lambda_l$, that is $\mathcal{P}_{\lambda_l}u = \sum_{m=1}^{m_l} \langle u, \phi_{l,m}\rangle_{L^2(\Omega)}\phi_{l,m}$, for $u \in L^2(\Omega)$.

In sections 2.3, 2.4 and 3.2, we consider the one-dimensional case. In particular $m_l = 1$ so that, for commodity, we simply use the notation ϕ_l instead of $\phi_{l,1}$. We also replace Δ by ∂_x^2.

In section 2.5, we use the notation $-\lambda_l^{X_1}$ (resp. $-\lambda_l^{X_2}$) to emphasize that this is the eigenvalues corresponding to the domain $\Omega = (0, X_1)$ (resp. $\Omega = (0, X_2)$), and we denote by $\phi_l^{X_1}$ (resp. $\phi_l^{X_2}$) a corresponding eigenfunction.

[3] In finite dimension all the notions of controllability are equivalent.

2. **A system of coupled heat equations.** We start by applying Theorem 1.1 to the operators

$$\mathcal{A} = \Delta + A^*, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega)^n \cap H_0^1(\Omega)^n,$$
$$\mathcal{C} = B^* 1_\gamma \partial_n, \quad \mathcal{D}(\mathcal{C}) = H^2(\Omega)^n \cap H_0^1(\Omega)^n.$$

By a perturbation argument we can check that $\mathcal{A}$ generates an analytic semigroup on $L^2(\Omega)^n$, has a compact resolvent and the system of root vectors of $\mathcal{A}^*$ is complete in $L^2(\Omega)^n$ (using, for instance, the Keldysh's perturbation theorem, see [21, Theorem 4.3, Chapter I, §4]), so that it satisfies the required hypothesis. On the other hand, the operator $\mathcal{C}$ is clearly relatively bounded with respect to $\mathcal{A}$.

Thus, system (1) is approximately controllable (at some time or at any time, see Remark 1.3) if and only if

$$\operatorname{Ker}(s - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_\gamma \partial_n) = \{0\}, \quad \forall s \in \mathbb{C}. \tag{5}$$

The next step is to describe the spectral elements of $\Delta + A^*$. Actually, it is not difficult to see that the spectrum of $\Delta + A^*$ is

$$\sigma(\Delta + A^*) = \{-\lambda_l + \theta_i\}_{l,i},$$

and its eigenspaces are

$$\operatorname{Ker}(s - (\Delta + A^*)) = \operatorname{span}\{w_{i,j}\phi_{l,m}\}_{\substack{i,j,l,m \\ -\lambda_l + \theta_i = s}}.$$

As we can see, the spectral structure of the operator $\Delta + A^*$ is somehow separated into a scalar differential part and a vectorial algebraic part. Moreover, the operator $\mathcal{C}$ we consider is $\mathcal{C} = B^* 1_\gamma \partial_n$, and $1_\gamma \partial_n$ acts on the scalar differential part while B^* acts on the vectorial algebraic part (recall that B is a constant matrix). In this particular situation we have good hopes to obtain an easier characterization than condition (5). This is what establish the results in the following sections.

Remark 2.1. We shall emphasize that the eigenvalues $-\lambda_l + \theta_i$ are not necessarily distinct. All along this work, for an eigenvalue $s \in \sigma(\Delta + A^*)$, we will denote by $l_1^s, \ldots, l_{r_s}^s$ and $i_1^s, \ldots, i_{r_s}^s$ (with possibly $r_s = 1$) all the distinct indices such that

$$s = -\lambda_{l_1^s} + \theta_{i_1^s} = \ldots = -\lambda_{l_{r_s}^s} + \theta_{i_{r_s}^s}.$$

Note that $r_s < +\infty$ since there is a finite number of θ_i. As as result, any $u \in \operatorname{Ker}(s - (\Delta + A^*))$ has a writing of the form

$$u = \sum_{k=1}^{r_s} \sum_{j,m} \alpha_{k,j,m} w_{i_k^s,j} \phi_{l_k^s,m},$$

for some $\alpha_{k,j,m} \in \mathbb{C}$.

Since we will always reason at s fixed, we will omit the dependence with respect to s during the proofs (for the sake of clarity), though we will keep this notation in the statements of the results.

2.1. **A sufficient condition in any dimension.** As noticed in Remark 2.1 it may happen that some eigenvalue s can be written as $s = -\lambda_l + \theta_i = -\lambda_{l'} + \theta_{i'}$ with $i' \neq i$ and $l' \neq l$. This phenomenon of "resonance" is a consequence of the coupling (the matrix A) and as a result is specific to the fact that we study a system, in contrast with a single equation. We will see that all the difficulties will precisely come from this point. This fact has been highlighted for the very first time in [14]. The following theorem shows that, when there is no phenomenon of resonance, the controllability is simply reduced to an algebraic condition, whatever the space dimension N and the control domain γ are.

Theorem 2.2. *Assume that*

$$r_s = 1, \quad \forall s \in \sigma\left(\Delta + A^*\right). \tag{6}$$

Then, the N-dimensional system

$$\begin{cases} \partial_t y - \Delta y &= Ay & \text{in } (0,T)\times\Omega, \\ y &= 1_\gamma B g & \text{on } (0,T)\times\partial\Omega, \\ y(0) &= y_0 & \text{in } \Omega, \end{cases}$$

is approximately controllable if and only if

$$\operatorname{Ker}\left(\theta_i - A^*\right) \cap \operatorname{Ker}\left(B^*\right) = \{0\}, \quad \forall i. \tag{7}$$

In general, the assumption (6) is not a necessary condition, except in some very particular cases, see Corollary 2.11 in section 2.4.

Remark 2.3. Condition (7) is nothing but the condition of Theorem 1.1 on the algebraic part of the system (see also Remark 1.4). We would also expect to require the similar condition concerning the scalar differential part, namely

$$\operatorname{Ker}\left(-\lambda_l - \Delta\right) \cap \operatorname{Ker}\left(1_\gamma \partial_n\right) = \{0\}, \quad \forall l, \tag{8}$$

but actually this condition is always fulfilled, see [20, Lemma], so that it is implicitly hidden in the theorem (and this will be used in the proof). This condition corresponds to the approximate controllability of the heat equation from the boundary.

Example 2.4. An easy but nontheless interesting consequence of Theorem 2.2 is when A^* has only one eigenvalue. In this case the assumption (6) is naturally satisfied. This permits for instance to easily prove that the N-dimensional cascade system

$$\begin{cases} \partial_t y - \Delta y &= \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} & 0 \end{pmatrix} y & \text{in } (0,T)\times\Omega, \\ y &= 1_\gamma \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} g & \text{on } (0,T)\times\partial\Omega, \\ y(0) &= y_0 & \text{in } \Omega, \end{cases}$$

is approximately controllable if and only if $a_{i,i-1} \neq 0$ for every i.

Proof of Theorem 2.2. Theorem 2.2 is a straightforward consequence of the following two lemma. The first lemma shows that condition (7) is always a necessary condition for the approximate controllability of system (1), while the second lemma shows that this condition is also enough to "control" the eigenvalues s for which

$r_s = 1$. Since we assume that there are only such eigenvalues, Theorem 2.2 will be proved. □

Lemma 2.5. *If system (1) is approximately controllable, then (7) holds.*

Lemma 2.6. *Assume that (7) holds. Then, for any eigenvalue $s \in \sigma(\Delta + A^*)$ such that $r_s = 1$, we have*

$$\operatorname{Ker}(s - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_\gamma \partial_n) = \{0\}.$$

Proof of Lemma 2.5. Let $w \in \operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*)$. Let $\lambda \in \sigma(\Delta)$. Taking any nonzero $\phi \in \operatorname{Ker}(\lambda - \Delta)$ we see that $u = \phi w$ belongs to $\operatorname{Ker}(\lambda + \theta_i - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_\gamma \partial_n)$, so that $u = 0$ by assumption, and thus also $w = 0$. □

Proof of Lemma 2.6. Let $s \in \sigma(\Delta + A^*)$ with $r_s = 1$, $u \in \operatorname{Ker}(s - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_\gamma \partial_n)$. Since $r_s = 1$, u writes

$$u = \sum_{j,m} \alpha_{j,m} w_{i_1,j} \phi_{l_1,m}$$

for some $\alpha_{j,m} \in \mathbb{C}$. Let us set $\beta_j = 1_\gamma \partial_n \left(\sum_m \alpha_{j,m} \phi_{l_1,m}\right) \in L^2(\partial\Omega)$ so that we have

$$B^* \left(\sum_j \beta_j w_{i_1,j} \right) = 0.$$

Since $\sum_j \beta_j w_{i_1,j} \in \operatorname{Ker}(\theta_{i_1} - A^*)$ we can use (7) to obtain $\sum_j \beta_j w_{i_1,j} = 0$. Using the linear independance of $\{w_{i_1,j}\}_j$ we deduce that $\beta_j = 0$ for every j, that is

$$1_\gamma \partial_n \left(\sum_m \alpha_{j,m} \phi_{l_1,m} \right) = 0, \quad \forall j.$$

Since $\sum_m \alpha_{j,m} \phi_{l_1,m} \in \operatorname{Ker}(-\lambda_{l_1} - \Delta)$, using now (8) gives $\sum_m \alpha_{j,m} \phi_{l_1,m} = 0$. Thanks to the linear independance of $\{\phi_{l_1,m}\}_m$ we conclude that $\alpha_{j,m} = 0$ for every j, m, that is $u = 0$. □

2.2. **Systems with as many controls as equations.** As a second result we recover the known fact (see [14, Theorem 5.3]) that we can control the system from the boundary if we put as many controls as equations. In this particular case, the coupling becomes inconsequential (the matrix A can even be $A = 0$, that is no coupling at all). This situation can be understood as n uncoupled equations with one control for each. This result has been obtained in [14] by means of a Carleman estimate but we provide here an alternative proof, which is also simpler in our case.

Theorem 2.7. *The N-dimensional system*

$$\begin{cases} \partial_t y - \Delta y = Ay & \text{in } (0,T) \times \Omega, \\ y = 1_\gamma B g & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

is approximately controllable if we assume that

$$\operatorname{Ker}(B^*) = \{0\}.$$

Proof. Let s be an eigenvalue of $\Delta + A^*$ and $u \in \mathrm{Ker}\,(s-(\Delta+A^*)) \cap \mathrm{Ker}\,(B^* 1_\gamma \partial_n)$. Then, u writes

$$u = \sum_{k=1}^{r} \sum_{j,m} \alpha_{k,j,m} w_{i_k,j} \phi_{l_k,m}$$

for some $\alpha_{k,j,m} \in \mathbb{C}$. Since $u \in \mathrm{Ker}\,(B^* 1_\gamma \partial_n)$ and $\mathrm{Ker}\,(B^*) = \{0\}$ by assumption, we have

$$\sum_{k,j} \left(\sum_m \alpha_{k,j,m} 1_\gamma \partial_n \phi_{l_k,m} \right) w_{i_k,j} = 0.$$

By the linear independence of $\{w_{i,j}\}_{i,j}$ we obtain

$$1_\gamma \partial_n \left(\sum_m \alpha_{k,j,m} \phi_{l_k,m} \right) = 0, \quad \forall k, \forall j.$$

Since $\sum_m \alpha_{k,j,m}\phi_{l_k,m} \in \mathrm{Ker}\,(-\lambda_{l_k} - \Delta)$ we deduce that $\sum_m \alpha_{k,j,m}\phi_{l_k,m} = 0$ (using (8)), and by the linear independence of $\{\phi_{l,m}\}_m$ it follows that $\alpha_{k,j,m} = 0$ for every k, j, m, that is $u = 0$. □

2.3. **The one-dimensional case.** The one-dimensional case is a very particular situation because the boundary is reduced to two points, $\{0\}$ and $\{L\}$, if $\Omega = (0, L)$. In particular, only three possibilities arise for γ, namely $\gamma = \{0\}$, $\gamma = \{L\}$ or $\gamma = \{0\} \cup \{L\}$. We will study these three cases.

The results of this section have already been obtained in [4], with another formulation though, and a different proof.

We start with the case $\gamma = \{0\}$ (we refer to the introduction for the notations):

Theorem 2.8. *The one-dimensional system*

$$\begin{cases} \partial_t y - \partial_x^2 y &= Ay \quad \text{in } (0,T)\times(0,L), \\ y &= 1_{\{0\}} Bg \quad \text{on } (0,T)\times\{0,L\}, \\ y(0) &= y_0 \quad \text{in } (0,L), \end{cases}$$

is approximately controllable if and only if, for every $s \in \sigma\left(\partial_x^2 + A^*\right)$, we have

$$\mathrm{rank}\left(B^* P_{i_1^s} | \cdots | B^* P_{i_{r_s}^s} \right) = \sum_{k=1}^{r_s} m_{i_k}.$$

Proof. Let $u \in \mathrm{Ker}\left(s - (\partial_x^2 + A^*)\right) \cap \mathrm{Ker}\left(B^* 1_{\{0\}} \partial_n\right)$, where $s \in \sigma\left(\partial_x^2 + A^*\right)$. We know that u writes

$$u = \sum_{k=1}^{r} \sum_{j,m} \alpha_{k,j,m} w_{i_k,j} \phi_{l_k}$$

for some $\alpha_{k,j,m} \in \mathbb{C}$ and we have

$$\sum_{k=1}^{r} \sum_{j=1}^{m_{i_k}} \alpha_{k,j} B^* w_{i_k,j} \phi'_{l_k}(0) = 0.$$

This implies that $\alpha_{k,j} = 0$ for every k, j if and only if the matrix

$$\left(\phi'_{l_1}(0) B^* P_{i_1} \quad | \quad \cdots \quad | \quad \phi'_{l_r}(0) B^* P_{i_r} \right)$$

has full rank, that is

$$\operatorname{rank}\left(\phi'_{l_1}(0)B^*P_{i_1} \quad \Big| \quad \cdots \quad \Big| \quad \phi'_{l_r}(0)B^*P_{i_r}\right) = \sum_{k=1}^{r} m_{i_k}.$$

To conclude it remains to observe that

$$\operatorname{rank}\left(\phi'_{l_1}(0)B^*P_{i_1} \quad \Big| \quad \cdots \quad \Big| \quad \phi'_{l_r}(0)B^*P_{i_r}\right) = \operatorname{rank}\left(B^*P_{i_1}|\cdots|B^*P_{i_r}\right)$$

since $\phi'_l(0) \neq 0$ for every l. □

The same result holds if we consider $\gamma = \{L\}$ instead of $\gamma = \{0\}$. When γ is the whole boundary, that is $\gamma = \{0\} \cup \{L\}$, we have the following characterization:

Theorem 2.9. *The one-dimensional system*

$$\begin{cases} \partial_t y - \partial_x^2 y &= Ay \quad \text{in } (0,T)\times(0,L), \\ y &= Bg \quad \text{on } (0,T)\times\{0,L\}, \\ y(0) &= y_0 \quad \text{in } (0,L), \end{cases}$$

is approximately controllable if and only if, for every $s \in \sigma\left(\partial_x^2 + A^*\right)$, we have

$$\operatorname{rank}\begin{pmatrix} \phi'_{l_1^s}(0)B^*P_{i_1^s} & \Big| & \cdots & \Big| & \phi'_{l_{r_s}^s}(0)B^*P_{i_{r_s}^s} \\ \phi'_{l_1^s}(L)B^*P_{i_1^s} & \Big| & & \Big| & \phi'_{l_{r_s}^s}(L)B^*P_{i_{r_s}^s} \end{pmatrix} = \sum_{k=1}^{r_s} m_{i_k}.$$

The proof is the same as the one of Theorem 2.8.

Remark 2.10. Further to these two theorems, we see that it may happen that system (1) is controllable with a control acting on both parts of the boundary whereas it is not controllable if the control only acts on one part. Indeed, let us consider on $\Omega = (0,\pi)$ the system described by

$$A = \begin{pmatrix} 0 & -4 \\ 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We recall that the eigenvalues of ∂_x^2 on $(0,\pi)$ are $-\lambda_l = -l^2$ and the corresponding eigenfunctions are $\phi_l(x) = \sqrt{\frac{2}{\pi}}\sin(lx)$. We can check that

$$\sigma(A^*) = \{\theta_1 = 1, \theta_2 = 4\},$$

$$\operatorname{Ker}(\theta_1 - A^*) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad \operatorname{Ker}(\theta_2 - A^*) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}.$$

Since $r_s = 1$ for every eigenvalue $s \in \sigma\left(\partial_x^2 + A^*\right)$ except $s = -\lambda_1 + \theta_1 = -\lambda_2 + \theta_2$, it is not difficult to check that the condition of Theorem 2.9 is fulfilled, whereas the one of Theorem 2.8 is not.

2.4. **Systems with only one control: $m = 1$.** Another interesting situation is when we try to control system (1) with only one control. This corresponds to $m = 1$, so that the matrix B is in fact a (column) vector.

Let us come back to the one-dimensional case. We can always assume that $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ for every i since it is a necessary condition (see Lemma 2.5). It is not difficult to see that this condition is equivalent to

$$\operatorname{rank} B^* P_i = m_i, \quad \forall i.$$

Thus, when B^* has now only one line, we necessarily have

$$m_i = 1.$$

In such a case, note also that P_i is reduced to $w_{i,1}$, so that $B^* P_i$ is a scalar, and $\operatorname{rank} B^* P_i = 1$ then simply means that this scalar is not zero. Thus, for every $s \in \sigma\left(\partial_x^2 + A^*\right)$, we have

$$\begin{cases} \operatorname{rank}\left(B^* P_{i_1^s} | \cdots | B^* P_{i_{r_s}^s}\right) = \operatorname{rank}(1|\cdots|1), \\ \displaystyle\sum_{k=1}^{r_s} m_{i_k} = r_s. \end{cases}$$

As a result, in this particular case which is $m = 1$, Theorem 2.8 becomes

Corollary 2.11. *Assume that $m = 1$. The one-dimensional system*

$$\begin{cases} \partial_t y - \partial_x^2 y &= Ay & \textit{in } (0,T) \times (0,L), \\ y &= 1_{\{0\}} Bg & \textit{on } (0,T) \times \{0,L\}, \\ y(0) &= y_0 & \textit{in } (0,L), \end{cases}$$

is approximately controllable if and only if the following two conditions hold:

1. $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ *for every i.*
2. *For every eigenvalue $s \in \sigma\left(\partial_x^2 + A^*\right)$ we have $r_s = 1$.*

This result is historically the first relevant difference between distributed and boundary controllability for parabolic systems (these properties are equivalent for the heat equation for instance). This has been proved in [14]. Moreover, this also shows that if this system is controllable with a boundary control then it is also controllable with a distributed control (recall that the distributed controllability of this system is characterized by only the first condition, see [2]). We insist on the fact that this is a result in dimension one; except in the framework of [1], the problem is open in higher space dimension.

We have a similar result for Theorem 2.9 when $m = 1$:

Corollary 2.12. *Assume that $m = 1$. The one-dimensional system*

$$\begin{cases} \partial_t y - \partial_x^2 y &= Ay & \textit{in } (0,T) \times (0,L), \\ y &= Bg & \textit{on } (0,T) \times \{0,L\}, \\ y(0) &= y_0 & \textit{in } (0,L), \end{cases}$$

is approximately controllable if and only if the following two conditions hold:

1. $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ *for every i.*

2. *For every eigenvalue* $s \in \sigma\left(\partial_x^2 + A^*\right)$*, either* $r_s = 1$*, either* $r_s = 2$ *with*

$$\operatorname{rank}\begin{pmatrix} \phi'_{l_1^s}(0) & \phi'_{l_2^s}(0) \\ \phi'_{l_1^s}(L) & \phi'_{l_2^s}(L) \end{pmatrix} = 2.$$

Finally, let us give a result in any dimension when $m = 1$.

Theorem 2.13. *Assume that* $m = 1$*. The* N*-dimensional system*

$$\left\{\begin{array}{rcll} \partial_t y & = & \Delta y + Ay & \text{in } (0,T)\times\Omega, \\ y & = & 1_\gamma B g & \text{on } (0,T)\times\partial\Omega, \\ y(0) & = & y_0 & \text{in } \Omega, \end{array}\right.$$

is approximately controllable if and only if the following two conditions hold:

1. $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ *for every* i*.*
2. *For every* $s \in \sigma(\Delta + A^*)$*, we have*

$$\left(\operatorname{Ker}\left(-\lambda_{l_1^s} - \Delta\right) + \ldots + \operatorname{Ker}\left(-\lambda_{l_{r_s}^s} - \Delta\right)\right) \cap \operatorname{Ker}(1_\gamma \partial_n) = \{0\}.$$

Note that the second condition is relevant only for s with $r_s > 1$, see (8).

Proof of Theorem 2.13. Let $s \in \sigma(\Delta + A^*)$ and let $u \in \operatorname{Ker}(s - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_\gamma \partial_n)$. We know that u writes

$$u = \sum_{k=1}^{r} \sum_{j,m} \alpha_{k,j,m} w_{i_k,j} \phi_{l_k,m}$$

for some, $\alpha_{k,j,m} \in \mathbb{C}$ and we have

$$1_\gamma \partial_n \left(\sum_k \sum_m \beta_{k,m} \phi_{l_k,m}\right) = 0,$$

where $\beta_{k,m} = \sum_j \alpha_{k,j,m} B^* w_{i_k,j}$. Since B^* is a row vector, $\beta_{k,m}$ is a scalar, so that we can use the second condition and obtain that $\sum_m \beta_{k,m}\phi_{l_k,m} = 0$ for every k. By the linear indepedence of $\{\phi_{l,m}\}_m$ we obtain that $\beta_{k,m} = 0$ for every k, m, that is

$$B^* \left(\sum_j \alpha_{k,j,m} w_{i_k,j}\right) = 0, \quad \forall k, m.$$

Using now the first condition this gives $\sum_j \alpha_{k,j,m} w_{i_k,j} = 0$ and it follows that $\alpha_{k,j,m} = 0$ for every k, j, m, that is $u = 0$.

Let us now show that these conditions are also necessary. We only prove it for the second condition since it is already known for the first one, see Lemma 2.5. Let $\phi = \phi_{l_1} + \ldots + \phi_{l_r}$, with $\phi_l \in \operatorname{Ker}(-\lambda_l - \Delta)$, be such that $1_\gamma \partial_n \phi = 0$. For every k, let w_{i_k} be any eigenvector of A^* associated with θ_{i_k}. We know that $B^* w_i$ is a scalar and $B^* w_i \neq 0$ thanks to the first condition (we have just recalled that it is a necessary condition). Thus, we can define

$$u = \frac{1}{B^* w_{i_1}} w_{i_1} \phi_{l_1} + \ldots + \frac{1}{B^* w_{i_r}} w_{i_r} \phi_{l_r}.$$

We can see that $u \in \operatorname{Ker}(s - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_\gamma \partial_n)$ so that $u = 0$ by assumption. It follows from the linear independence of $\{w_i\}_i$ that $\phi_{l_k} = 0$ for every k, that is $\phi = 0$. ∎

2.5. **Analysis on a particular geometry.** In this section we still consider system (1) but the domain Ω is now a rectangle

$$\Omega = (0, X_1) \times (0, X_2).$$

We denote the faces of our rectangle by γ_L, γ_R, γ_T and γ_B, as on Figure 1:

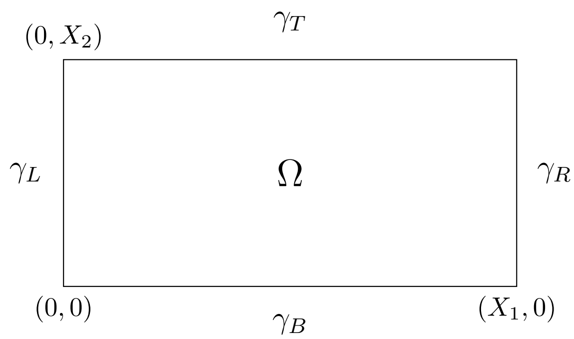

FIGURE 1. Domain Ω for section 2.5.

The goal of this section is to prove several results about the boundary controllability of system (1), by discussing on the geometric position of γ.

Theorem 2.14. *If $\gamma = \gamma_L$, then system (1) is approximately controllable if and only if so is the following one-dimensional system*

$$\begin{cases} \partial_t y - \partial_{x_1}^2 y = Ay & \text{in } (0,T) \times (0, X_1), \\ y = 1_{\{0\}} Bg & \text{on } (0,T) \times \{0, X_1\}, \\ y(0) = y_0 & \text{in } (0, X_1). \end{cases} \tag{9}$$

If $\gamma = \gamma_L \cup \gamma_R$ then system (1) is approximately controllable if and only if so is the following one-dimensional system

$$\begin{cases} \partial_t y - \partial_{x_1}^2 y = Ay & \text{in } (0,T) \times (0, X_1), \\ y = Bg & \text{on } (0,T) \times \{0, X_1\}, \\ y(0) = y_0 & \text{in } (0, X_1). \end{cases}$$

We recall that the controllability of these one-dimensional systems has been studied in sections 2.3 and 2.4.

For the heat equation, a similar result has been established in [13, 23] for the null-controllability when γ is one of the faces of $\partial\Omega$.

We consider next the case of two consecutive faces, with for instance $\gamma = \gamma_L \cup \gamma_T$.

Theorem 2.15. *Assume that $\mathrm{Ker}\,(\theta_i - A^*) \cap \mathrm{Ker}\,(B^*) = \{0\}$ for every i. If $\gamma = \gamma_L \cup \gamma_T$ and $n = 2$, then system (1) is approximately controllable.*

The geometry of γ (including two different directions γ_L and γ_T) is such that in some sense it "creates" an additional control. Thus, everything happens as if we had two controls for two equations and we can expect the controllability to hold, as it is showed in section 2.2. Theorem 2.14 shows that this is not true if we pick two parallel faces $\gamma = \gamma_L \cup \gamma_R$. When more equations are considered, the following counter-example strengthen this point of view.

Proposition 2.16. *Even if* $\gamma = \gamma_L \cup \gamma_T$ *and* $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ *for every* i*, system (1) may be not approximately controllable when* $n > 2$*.*

Remark 2.17. It is worth mentioning that, in all the previous statements, we can replace γ_L (resp. γ_R, γ_T, γ_B) by a nonempty open part of it. This is easily seen in the following proofs by using the analyticity of the one-dimensional eigenfunctions of $\partial^2_{x_2}$ (or $\partial^2_{x_1}$).

The main ingredient that will make the proofs work is the following. The (not necessarily distinct) eigenvalues of $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ on $(0, X_1) \times (0, X_2)$ are

$$-\Lambda_{p,q} = -\lambda_p^{X_1} - \lambda_q^{X_2},$$

and the corresponding eigenfunctions are

$$\Phi_{p,q}(x_1, x_2) = \phi_p^{X_1}(x_1)\phi_q^{X_2}(x_2), \quad x_1 \in (0, X_1), x_2 \in (0, X_2).$$

In this case the dimension m_l of the eigenspace of Δ associated with $-\lambda_l$ is exactly the number of distinct couples of indices (p, q) such that $\Lambda_{p,q} = \lambda_l$. We denote by $(p_l^1, q_l^1), \ldots, (p_l^{m_l}, q_l^{m_l})$ all such indices. Note that for every m we necessarily have

$$p_l^m \neq p_l^{m'} \quad \text{and} \quad q_l^m \neq q_l^{m'}, \quad \forall m' \neq m. \tag{10}$$

Indeed, it follows from the definition of these indices and the form of $\Lambda_{p,q}$ that, if $p_l^m = p_l^{m'}$, then we also have $q_l^m = q_l^{m'}$, which is excluded by definition.

Proof of Theorem 2.14. Let us consider the case $\gamma = \gamma_L$; the proof for $\gamma = \gamma_L \cup \gamma_R$ relies on the same kind of arguments. We also only prove that, if system (9) is approximately controllable, then so is system (1), the converse being easier.

Let $s \in \sigma(\Delta + A^*)$ and $u \in \operatorname{Ker}(s - (\Delta + A^*)) \cap \operatorname{Ker}(B^* 1_{\gamma_L} \partial_n)$. With the notations previously introduced u then writes

$$u = \sum_{k=1}^{r} \sum_{m=1}^{m_{l_k}} \left(\sum_j \alpha_{k,j,m} w_{i_k,j} \right) \phi^{X_1}_{p^m_{l_k}} \phi^{X_2}_{q^m_{l_k}},$$

for some $\alpha_{k,j,m} \in \mathbb{C}$, and we have

$$\sum_{k=1}^{r} \sum_{m=1}^{m_{l_k}} \beta_{k,m} \phi^{X_2}_{q^m_{l_k}}(x_2) = 0, \quad \forall x_2 \in (0, X_2), \tag{11}$$

where we have set

$$\beta_{k,m} = -\gamma_{k,m} \left(\phi^{X_1}_{p^m_{l_k}} \right)'(0), \quad \gamma_{k,m} = B^* \left(\sum_j \alpha_{k,j,m} w_{i_k,j} \right).$$

Note that we can always assume that $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ for every i since it is a necessary condition (see Lemma 2.5) for both systems (1) and (9). As a result, to prove that $u = 0$ it is equivalent to show that $\gamma_{k,m} = 0$ (or $\beta_{k,m} = 0$) for every k, m.

For convenience we assume that $r = 2$. Thus (11) becomes

$$\sum_{m=1}^{m_{l_1}} \beta_{1,m} \phi^{X_2}_{q^m_{l_1}}(x_2) + \sum_{m=1}^{m_{l_2}} \beta_{2,m} \phi^{X_2}_{q^m_{l_2}}(x_2) = 0, \quad \forall x_2 \in (0, X_2).$$

Using the linear independence of $\left\{\phi_q^{X_2}\right\}_q$ in $L^2(0, X_2)$, two cases may happen. For some given m, if $q^m_{l_1} \neq q^{m'}_{l_2}$ for every m' then, taking also (10) into account, we obtain $\beta_{1,m}=0$. On the other hand, if there exists m' such that $q^m_{l_1}=q^{m'}_{l_2}$, then this m' is unique thanks to (10) and we obtain that $\beta_{1,m}+\beta_{2,m'}=0$, that is

$$-\left(\gamma_{1,m}\phi^{X_1}_{p^m_{l_1}}+\gamma_{2,m'}\phi^{X_1}_{p^{m'}_{l_2}}\right)'(0)=0.$$

Since $q^m_{l_1}=q^{m'}_{l_2}$ we have

$$-\lambda^{X_1}_{p^m_{l_1}}+\theta_{i_1}=-\lambda^{X_1}_{p^{m'}_{l_2}}+\theta_{i_2},$$

and the assumption that system (9) is approximately controllable permits to conclude that $\gamma_{1,m}=\gamma_{2,m'}=0$.

Thus, in both situations $\gamma_{1,m}=0$, and it follows that $\gamma_{2,m}=0$ (when $r>2$ we reason by induction). □

Proof of Theorem 2.15. Since we assume that $n=2$, the matrix A^* has at most two distinct eigenvalues. If A^* has only one eigenvalue then we already know that the system is approximately controllable, see Example 2.4 in section 2.1. Let us then assume that A^* has two distinct eigenvalues

$$\theta_{i_1}\neq\theta_{i_2}. \tag{12}$$

With the same notations as in the proof of Theorem 2.14, let us show that is it not possible to have

$$\begin{cases}\displaystyle\sum_{m=1}^{m_{l_1}}\beta_{1,m}\phi^{X_2}_{q^m_{l_1}}+\sum_{m=1}^{m_{l_2}}\beta_{2,m}\phi^{X_2}_{q^m_{l_2}}=0,\\ \displaystyle\sum_{m=1}^{m_{l_1}}\delta_{1,m}\phi^{X_1}_{p^m_{l_1}}+\sum_{m=1}^{m_{l_2}}\delta_{2,m}\phi^{X_1}_{p^m_{l_2}}=0,\end{cases}$$

with $\gamma_{k,m}\neq0$ for every k,m, where

$$\delta_{k,m}=\gamma_{k,m}\left(\phi^{X_2}_{q^m_{l_k}}\right)'(X_2).$$

From the first equation we see that the sets $\{q^m_{l_1}\}_{1\le m\le m_{l_1}}$ and $\left\{q^{m'}_{l_2}\right\}_{1\le m'\le m_{l_2}}$ are in bijection. Indeed, if there exists m such that $q^m_{l_1}\neq q^{m'}_{l_2}$ for every m' then, using the linear independence of $\{\phi^{X_2}_q\}_q$ in $L^2(0,X_2)$, we obtain $\beta_{1,m}=\gamma_{1,m}=0$. Since the same fact holds for the second equation, the sets $\{p^m_{l_1}\}_{1\le m\le m_{l_1}}$ and $\left\{p^{m'}_{l_2}\right\}_{1\le m'\le m_{l_2}}$ are also in bijection.

As a consequence, denoting $M=m_{l_1}=m_{l_2}$, we have

$$\sum_{m=1}^{M}\lambda^{X_2}_{q^m_{l_1}}=\sum_{m'=1}^{M}\lambda^{X_2}_{q^{m'}_{l_2}},\quad \sum_{m=1}^{M}\lambda^{X_1}_{p^m_{l_1}}=\sum_{m'=1}^{M}\lambda^{X_1}_{p^{m'}_{l_2}},$$

so that

$$\sum_{m=1}^{M}\Lambda_{p^m_{l_1},q^m_{l_1}}=\sum_{m'=1}^{M}\Lambda_{q^{m'}_{l_2},q^{m'}_{l_2}}.$$

Let us denote by S this common value. Since $-\Lambda_{p^m_{l_1},q^m_{l_1}}+\theta_{i_1}=-\Lambda_{q^m_{l_2},q^m_{l_2}}+\theta_{i_2}$ for every m, if we sum we obtain

$$-S+M\theta_{i_1}=-S+M\theta_{i_2},$$

and thus

$$\theta_{i_1} = \theta_{i_2},$$

a contradiction with our assumption (12). □

Proof of Proposition 2.16. We provide an example of system with 4 equations for which the condition $\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}$ holds for every i and that is not approximately controllable on $\gamma = \gamma_L \cup \gamma_T$. This example can easily be generalized to the case $n > 4$.

We take $X_1 = X_2 = \pi$, so that the eigenvalues of Δ are simply

$$-\Lambda_{p,q} = -p^2 - q^2,$$

and we choose

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -10 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can check that

$$\sigma(A^*) = \{\theta_1 = -8, \theta_2 = -5, \theta_3 = 3, \theta_4 = 0\}.$$

$$\operatorname{Ker}(\theta_i - A^*) \cap \operatorname{Ker}(B^*) = \{0\}, \quad i = 1, 2, 3, 4.$$

Now, observe that we have the relation

$$-\Lambda_{1,1} + \theta_1 = -\Lambda_{2,1} + \theta_2 = -\Lambda_{2,3} + \theta_3 = -\Lambda_{1,3} + \theta_4 = -10.$$

In view of this relation we define

$$u = \phi_{1,1} - \frac{1}{2}\phi_{2,1} + \frac{1}{6}\phi_{2,3} - \frac{1}{3}\phi_{1,3}.$$

Clearly $u \neq 0$. Let us show that however $\partial_n u = 0$ on $\gamma_L \cup \gamma_T$, which will prove Proposition 2.16 thanks to Theorem 2.13. Taking into account that $\left(\phi_p^{X_1}\right)'(0) = p\sqrt{\frac{2}{\pi}}$, for $x_2 \in (0, \pi)$ we have

$$-\partial_{x_1} u(0, x_2) = -\underbrace{\left(\left(\phi_1^{X_1}\right)'(0) - \frac{1}{2}\left(\phi_2^{X_1}\right)'(0)\right)}_{=0}\phi_1^{X_2}(x_2) - \underbrace{\left(\frac{1}{6}\left(\phi_2^{X_1}\right)'(0) - \frac{1}{3}\left(\phi_1^{X_1}\right)'(0)\right)}_{=0}\phi_3^{X_2}(x_2),$$

so that indeed $\partial_n u = 0$ on γ_L. In the same way, for $x_1 \in (0, \pi)$ we have

$$\partial_{x_2} u(x_1, \pi) = \phi_1^{X_1}(x_1)\underbrace{\left(\left(\phi_1^{X_2}\right)'(\pi) - \frac{1}{3}\left(\phi_3^{X_2}\right)'(\pi)\right)}_{=0} + \phi_2^{X_1}(x_1)\underbrace{\left(-\frac{1}{2}\left(\phi_1^{X_2}\right)'(\pi) + \frac{1}{6}\left(\phi_3^{X_2}\right)'(\pi)\right)}_{=0},$$

and thus $\partial_n u = 0$ also on γ_T. □

3. **A cascade system coupled by a first order term.** We now turn out to the results concerning the second system

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t y_2 - \Delta y_2 = G(x)\cdot\nabla y_1 + a(x) y_1 & \text{in } (0,T)\times\Omega, \\ y_1 = 1_\gamma g, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega. \end{cases}$$

As mentionned in the introduction, it is known that this system is approximately controllable at time T if and only if its adjoint system

$$\begin{cases} \partial_t z_1 - \Delta z_1 = -G(x)\cdot\nabla z_2 + (a(x) - \operatorname{div} G(x)) z_2 & \text{in } (0,T)\times\Omega, \\ \partial_t z_2 - \Delta z_2 = 0 & \text{in } (0,T)\times\Omega, \\ z_1 = 0, \quad z_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ z_1(0) = z_{1,0}, \quad z_2(0) = z_{2,0} & \text{in } \Omega, \end{cases}$$

has the unique continuation property

$$\forall z_{1,0}, z_{2,0} \in H^1_0(\Omega), \quad \Big(1_\gamma \partial_n z_1(t) = 0 \text{ for a.e. } t\in(0,T)\Big) \Longrightarrow z_{1,0} = z_{2,0} = 0.$$

For commodity, let us denote

$$\mathcal{Q} = -G(x)\cdot\nabla + (a(x) - \operatorname{div} G(x)).$$

As before we apply Theorem 1.1, this time to the operators

$$\mathcal{A} = \begin{pmatrix} \Delta & \mathcal{Q} \\ 0 & \Delta \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = H^2(\Omega)^2 \cap H^1_0(\Omega)^2,$$

$$\mathcal{C} = \begin{pmatrix} 1_\gamma \partial_n & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{C}) = H^2(\Omega)^2 \cap H^1_0(\Omega)^2.$$

Again, by using a perturbation argument we can check that $\mathcal{A}$ generates analytic semigroup and indeed satisfies the hypothesis of Theorem 1.1. The operator $\mathcal{C}$ is of the same kind as for the first system we studied. As a consequence, this system is approximately controllable if and only if

$$\operatorname{Ker}\begin{pmatrix} s-\Delta & -\mathcal{Q} \\ 0 & s-\Delta \end{pmatrix} \cap \operatorname{Ker}\begin{pmatrix} 1_\gamma\partial_n & 0 \end{pmatrix} = \{0\}, \quad \forall s\in\mathbb{C}. \tag{13}$$

It is not difficult to see that the spectrum of $\mathcal{A}$ is

$$\sigma(\mathcal{A}) = \{-\lambda_l\}_l,$$

and that its eigenspaces can be decomposed as follows:

$$\operatorname{Ker}(-\lambda_l - \mathcal{A}) = U_l \oplus V_l,$$

with

$$U_l = \left\{\begin{pmatrix} u \\ 0 \end{pmatrix}\right\}_{u\in\operatorname{Ker}(-\lambda_l-\Delta)}, \quad V_l = \left\{\begin{pmatrix} \mathcal{S}_l\mathcal{Q}v \\ v \end{pmatrix}\right\}_{v\in\operatorname{Ker}(-\lambda_l-\Delta)\cap\operatorname{Ker}\left(\mathcal{P}_{\lambda_l}\mathcal{Q}\right)},$$

where $\mathcal{S}_l : f \in \mathrm{Ker}\,(\mathcal{P}_{\lambda_l}) \longmapsto u \in \mathrm{Ker}\,(\mathcal{P}_{\lambda_l})$ with u the unique solution (in $\mathrm{Ker}\,(\mathcal{P}_{\lambda_l})$) of the equation $(-\lambda_l - \Delta)u = f$.

3.1. **A sufficient condition in any dimension.** The following theorem is, in some sense, the analogue of Theorem 2.2 in section 2.1. This also recovers [19, Theorem 1.5].

Theorem 3.1. *Assume that*

$$\mathrm{Ker}\,(-\lambda_l - \Delta) \cap \mathrm{Ker}\,(\mathcal{P}_{\lambda_l}\mathcal{Q}) = \{0\}\,, \quad \forall l. \tag{14}$$

Then, the N-dimensional system

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0 & \text{in } (0,T)\times\Omega,\\ \partial_t y_2 - \Delta y_2 = G(x)\cdot\nabla y_1 + a(x)y_1 & \text{in } (0,T)\times\Omega,\\ y_1 = 1_\gamma g, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega,\\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega, \end{cases}$$

is approximately controllable.

Proof. The assumption (14) means that $V_l = \{0\}$ for every l, so that $\mathrm{Ker}\,(-\lambda_l - \mathcal{A}) = U_l$ for every l. Thus, any $w \in \mathrm{Ker}\,(-\lambda_l - \mathcal{A}) \cap \mathrm{Ker}\,(\mathcal{C})$ writes

$$w = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

for some $u \in \mathrm{Ker}\,(-\lambda_l - \Delta)$ and satisfies $Cw = 1_\gamma \partial_n u = 0$. This implies $u = 0$ (see (8)) and thus also $w = 0$. ☐

Remark 3.2. Condition (14) can be reformulated into the following rank condition:

$$\mathrm{rank}\begin{pmatrix} \langle \mathcal{Q}\phi_{l,1}, \phi_{l,1}\rangle_{L^2(\Omega)} & \cdots & \langle \mathcal{Q}\phi_{l,1}, \phi_{l,m_l}\rangle_{L^2(\Omega)} \\ \vdots & & \vdots \\ \langle \mathcal{Q}\phi_{l,m_l}, \phi_{l,1}\rangle_{L^2(\Omega)} & \cdots & \langle \mathcal{Q}\phi_{l,m_l}, \phi_{l,m_l}\rangle_{L^2(\Omega)} \end{pmatrix} = m_l, \quad \forall l. \tag{15}$$

3.2. **Complete caracterization in dimension one.** As for Corollary 2.11 in section 2.4, condition (14) turns out to be also necessary in dimension one:

Theorem 3.3. *The one-dimensional system*

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{in } (0,T)\times(0,L),\\ \partial_t y_2 - \partial_x^2 y_2 = G(x)\partial_x y_1 + a(x)y_1 & \text{in } (0,T)\times(0,L),\\ y_1 = 1_{\{0\}} g, \quad y_2 = 0 & \text{on } (0,T)\times\{0,L\}\,,\\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } (0,L), \end{cases} \tag{16}$$

is approximately controllable if and only if

$$\int_0^L \left(-\frac{1}{2}G'(x) + a(x)\right) |\phi_l(x)|^2\, dx \neq 0, \quad \forall l. \tag{17}$$

Example 3.4. For $a = 0$ and G constant, the corresponding system is not approximately controllable.

Proof of Theorem 3.3. From Theorem 3.1 we know that (14) is a sufficient condition. Let us prove that it is also necessary in dimension one. To this end, assume that (14) does not hold. Then, for some l, there exists at least one eigenfunction of $\mathcal{A}$ associated with $-\lambda_l$ in U_l and another one in V_l, say

$$\begin{pmatrix} u \\ 0 \end{pmatrix} \in U_l, \quad w = \begin{pmatrix} \mathcal{S}_l \mathcal{Q} v \\ v \end{pmatrix} \in V_l.$$

If $Cw = 0$, that is $(\mathcal{S}_l \mathcal{Q} v)'(0) = 0$, then the approximate controllability condition (13) already fails. On the other hand, if $(\mathcal{S}_l \mathcal{Q} v)'(0) \neq 0$, then condition (13) also fails because of the following relation

$$\left(\frac{1}{u'(0)} u - \frac{1}{(\mathcal{S}_l \mathcal{Q} v)'(0)} \mathcal{S}_l \mathcal{Q} v \right)'(0) = 0.$$

As a result, (14) is a necessary and sufficient condition in dimension one. To conclude it remains to observe that, since $m_l = 1$ for every l ($N = 1$), condition (15) now reads as

$$\langle \mathcal{Q} \phi_l, \phi_l \rangle_{L^2(\Omega)} \neq 0, \quad \forall l,$$

which gives condition (17) after an integration by part on the gradient term. □

4. **Further results: Distributed controllability.** All along this work we were interested in the boundary controllability problem but let us mention that the method also works for distributed controllability. For instance, we can recover the result of [2] concerning system (1). We can also obtain the following result:

Theorem 4.1. *Let ω be a nonempty open subset of Ω. Assume that Ω is connected and G and a are real analytic functions in Ω. Then, the N-dimensional system*

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 1_\omega g & \text{in } (0,T) \times \Omega, \\ \partial_t y_2 - \Delta y_2 = G(x) \cdot \nabla y_1 + a(x) y_1 & \text{in } (0,T) \times \Omega, \\ y_1 = 0, \quad y_2 = 0 & \text{on } (0,T) \times \partial\Omega, \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega, \end{cases}$$

is approximately controllable if and only if

$$\operatorname{Ker}(-\lambda_l - \Delta) \cap \operatorname{Ker}(\mathcal{Q}) = \{0\}, \quad \forall l, \tag{18}$$

where we recall that $\mathcal{Q} = -G(x) \cdot \nabla + (a(x) - \operatorname{div} G(x))$.

To the author knowledge [8] and [16] are the only works for the distributed controllability of this system. However in these papers, even for $a = 0$ and G constant, additional assumptions are needed. Indeed, for this case the system is null-controllable in dimension one (consequence of [16, Theorem 4]) or in any dimension but with a geometric condition on ω ([8, Theorem 1.1]).

Proof of Theorem 4.1. This time the observation operator is $\mathcal{C} = \begin{pmatrix} 1_\omega & 0 \end{pmatrix}$ (it is a bounded operator on $L^2(\Omega)^2$). Let $w \in \operatorname{Ker}(-\lambda_l - \mathcal{A}) \cap \operatorname{Ker}(\mathcal{C})$. Thus, w writes

$$w = \begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{S}_l \mathcal{Q} v \\ v \end{pmatrix}$$

for some $u \in \mathrm{Ker}\,(-\lambda_l - \Delta)$ and $v \in \mathrm{Ker}\,(-\lambda_l - \Delta) \cap \mathrm{Ker}\,(\mathcal{P}_{\lambda_l}\mathcal{Q})$, and satisfies

$$1_\omega\,(u + \mathcal{S}_l\mathcal{Q}v) = 0. \tag{19}$$

Since v is an analytic function, so is $\mathcal{Q}v$. Thus, $\mathcal{S}_l\mathcal{Q}v$ is an analytic function as solution of an elliptic partial differential equation with analytic data (see for instance [18, Theorem 7.5.1]). Note that u is also analytic. Thus, (19) is equivalent to

$$u + \mathcal{S}_l\mathcal{Q}v = 0.$$

This implies that $u = 0$ since $u = -\mathcal{S}_l\mathcal{Q}v \in \mathrm{Ker}\,(\mathcal{P}_{\lambda_l})$ and $u \in \mathrm{Ker}\,(-\lambda_l - \Delta) = \mathrm{Im}\,(\mathcal{P}_{\lambda_l})$. Thus, $\mathcal{S}_l\mathcal{Q}v = -u = 0$ and it follows that $\mathcal{Q}v = 0$ (see the definition of $\mathcal{S}_l$). This implies $v = 0$ if and only if (18) holds. □

Example 4.2. Let us illustrate this result with $a = 0$ and $G \neq 0$ constant. This means that we take $\mathcal{Q} = -G \cdot \nabla$. We can verify that this operator $\mathcal{Q}$ satisfies (18) since we actually have the stronger property

$$\forall u \in H_0^1(\Omega), \quad G \cdot \nabla u = 0 \text{ in } \Omega \Longrightarrow u = 0.$$

Indeed, set $v(x) = e^{G\cdot x}u(x)$. We have $v \in H_0^1(\Omega)$ and (using the hypothesis on u)

$$G \cdot \nabla v = |G|^2\, v.$$

Multiplying this equality by v and integrating by parts we obtain

$$|G|^2 \int_\Omega |v(x)|^2\, dx = 0.$$

This implies that $v = 0$ and thus also $u = 0$.

As a result, we can apply Theorem 4.1 and obtain that the N-dimensional system

$$\begin{cases} \partial_t y_1 - \Delta y_1 = 1_\omega g & \text{in } (0,T)\times\Omega, \\ \partial_t y_2 - \Delta y_2 = G\cdot\nabla y_1 & \text{in } (0,T)\times\Omega, \\ y_1 = 0, \quad y_2 = 0 & \text{on } (0,T)\times\partial\Omega, \\ y_1(0) = y_{1,0}, \quad y_2(0) = y_{2,0} & \text{in } \Omega, \end{cases}$$

is approximately controllable.

Appendix A. Proof of Theorem 1.1. For the sake of completeness we give here the proof of Theorem 1.1. We recall that this proof is just adapted from the one in [12] in order to deal with relatively bounded observation operators.

Let us recall the notations and assumptions. H and U are complex Hilbert spaces, $\mathcal{A} : \mathcal{D}\,(\mathcal{A}) \subset H \longrightarrow H$ generates a strongly continuous semigroup on H, has a compact resolvent, the system of root vectors of $\mathcal{A}^*$ is complete in H, and $\mathcal{C} : \mathcal{D}\,(\mathcal{C}) \subset H \longrightarrow U$ is relatively bounded with respect to $\mathcal{A}$.

We denote by $\rho(\mathcal{A})$ the resolvent set of our closed linear operator $\mathcal{A}$ and, for $\lambda \in \rho(\mathcal{A})$, $\mathcal{R}\,(\lambda;\mathcal{A}) = (\lambda - \mathcal{A})^{-1}$ the resolvent operator.

Since $\mathcal{A}$ has a compact resolvent, its spectrum $\sigma\,(\mathcal{A}) = \mathbb{C}\backslash\rho(\mathcal{A})$ consists in a sequence of isolated points, say $\{\mu_j\}_j$. In particular $\rho(\mathcal{A})$ is path connected. We have $\sigma\,(\mathcal{A}^*) = \overline{\sigma\,(\mathcal{A})} = \{\overline{\mu_j}\}_j$.

Let now $C_j \subset \rho(\mathcal{A})$ be a positive-oriented small circle enclosing μ_j and such that no other eigenvalue than μ_j lies inside this circle. For every j we define the spectral projection

$$\begin{array}{rccl} \mathcal{P}_{\mu_j}: & H & \longrightarrow & H \\ & u & \longmapsto & \dfrac{1}{2\pi i}\displaystyle\int_{C_j} \mathcal{R}\left(\xi;\mathcal{A}\right) u \, d\xi. \end{array}$$

The operator $\mathcal{P}_{\mu_j}$ is a bounded linear operator and one can prove that the range of this operator is exactly the root subspace of $\mathcal{A}$ associated with μ_j, i.e. $\operatorname{Ker}\left(\mu_j - \mathcal{A}\right)^{\tau_j}$, where τ_j is the smallest indice k such that $\operatorname{Ker}\left(\mu_j - \mathcal{A}\right)^{k+1} = \operatorname{Ker}\left(\mu_j - \mathcal{A}\right)^{k}$. For a proof of this fact we refer to [10, 2 Lemma, Chapter XIX]. A computation shows that

$$\mathcal{P}^*_{\mu_j} = \frac{1}{2\pi i}\int_{\overline{C_j}} \mathcal{R}\left(\xi;\mathcal{A}^*\right) \, d\xi,$$

where $\overline{C_j}$ is the circle centered in $\overline{\mu_j}$ with the same radius as C_j. Since there are no eigenvalue of $\mathcal{A}^*$ except $\overline{\mu_j}$ inside the circle $\overline{C_j}$, the range of this operator is exactly the root subspace of $\mathcal{A}^*$ associated with $\overline{\mu_j}$.

Let us now recall some properties of semigroups. Since $\mathcal{A}$ generates a strongly continuous semigroup $\mathcal{S}(t)$ on H, we know that there exists $M > 0$ and $\omega_0 \in \mathbb{R}$ such that

$$||\mathcal{S}(t)||_{\mathcal{L}(H)} \leq M e^{\omega_0 t}, \quad \forall t \geq 0.$$

Moreover, for every $z_0 \in \mathcal{D}\left(\mathcal{A}\right)$, we have $\mathcal{S}(t)z_0 \in \mathcal{D}\left(\mathcal{A}\right)$ with $\mathcal{A}\mathcal{S}(t)z_0 = \mathcal{S}(t)\mathcal{A}z_0$ and the map $t \in [0,+\infty) \longmapsto \mathcal{S}(t)z_0 \in \mathcal{D}\left(\mathcal{A}\right)$ is continuous. Finally, the resolvent set $\rho(A)$ contains the halfplane $\{\lambda \in \mathbb{C} \,|\, \mathfrak{Re}\,\lambda > \omega_0\}$. For a proof of these facts we refer to [11, Proposition 5.5, Chapter I], [11, Lemma 1.3, Chapter II] and [11, Theorem 1.10, Chapter II], respectively.

Lemma A.1 (Corollary 2.2 of [12])**.** *Let $z_0 \in \mathcal{D}\left(\mathcal{A}\right)$ be fixed. The three following properties are equivalent:*

1. *$\mathcal{C}\mathcal{S}(t)z_0 = 0$ for a.e. $t \in (0,+\infty)$.*
2. *$\mathcal{C}\mathcal{R}\left(\lambda;\mathcal{A}\right) z_0 = 0$ for every $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\,\lambda > \omega_0$.*
3. *$\mathcal{C}\mathcal{R}\left(\lambda;\mathcal{A}\right) z_0 = 0$ for every $\lambda \in \rho(\mathcal{A})$.*

Proof. Recall that the resolvent of an operator can be represented as the Laplace transform of the semigroup it generates for $\mathfrak{Re}\,\lambda > \omega_0$ (see for instance [11, Theorem 1.10, Chapter II]):

$$\begin{aligned} \mathcal{R}\left(\lambda;\mathcal{A}\right) z_0 &= \int_0^{+\infty} e^{-\lambda t}\mathcal{S}(t)z_0 \, dt \\ &= \lim_{j\to+\infty} \int_0^{j} e^{-\lambda t}\mathcal{S}(t)z_0 \, dt \quad \text{(limit in } H). \end{aligned}$$

Actually, this limit can also be considered in the sense of $\mathcal{D}\left(\mathcal{A}\right)$. Indeed,

$$\int_0^{j} e^{-\lambda t}\mathcal{S}(t)z_0 \, dt \in \mathcal{D}\left(\mathcal{A}\right),$$

with

$$\mathcal{A}\int_0^{j} e^{-\lambda t}\mathcal{S}(t)z_0 \, dt = \int_0^{j} e^{-\lambda t}\mathcal{A}\mathcal{S}(t)z_0 \, dt = \int_0^{j} e^{-\lambda t}\mathcal{S}(t)\mathcal{A}z_0 \, dt,$$

$$\xrightarrow[j\to+\infty]{} \mathcal{R}\left(\lambda;\mathcal{A}\right)\mathcal{A}z_0.$$

Since $\mathcal{C}$ is bounded on $\mathcal{D}(\mathcal{A})$ we obtain

$$\mathcal{C}\mathcal{R}(\lambda;\mathcal{A})\, z_0 = \int_0^{+\infty} e^{-\lambda t}\mathcal{C}\mathcal{S}(t) z_0 \, dt, \quad \mathfrak{Re}\,\lambda > \omega_0.$$

It is now clear that 1. implies 2. while the converse follows from the injectivity of the Laplace transform.

The remaining equivalence is a consequence of the analytic continuation of the resolvent. □

Lemma A.2 (Proposition 3.1 of [12])**.** *Let $z_0 \in \mathcal{D}(\mathcal{A})$ be fixed. If the third point of Lemma A.1 holds, then $\mathcal{C}\mathcal{R}(\lambda;\mathcal{A})\,\mathcal{P}_{\mu_j} z_0 = 0$ for every $\lambda \in \rho(\mathcal{A})$ and every j.*

Proof. Let $\lambda \in \rho(\mathcal{A})$ lies outside the circle C_j.

The first resolvent equation $(\lambda - \xi)\mathcal{R}(\lambda;\mathcal{A})\,\mathcal{R}(\xi;\mathcal{A}) = \mathcal{R}(\xi;\mathcal{A}) - \mathcal{R}(\lambda;\mathcal{A})$ gives

$$\begin{aligned}
\mathcal{R}(\lambda;\mathcal{A})\,\mathcal{P}_{\mu_j} z_0 &= \mathcal{R}(\lambda;\mathcal{A})\,\frac{1}{2\pi i}\int_{C_j} \mathcal{R}(\xi;\mathcal{A})\, z_0 \, d\xi \\
&= \frac{1}{2\pi i}\int_{C_j} \mathcal{R}(\lambda;\mathcal{A})\,\mathcal{R}(\xi;\mathcal{A})\, z_0\, d\xi \\
&= -\frac{1}{2\pi i}\int_{C_j} \frac{\mathcal{R}(\xi;\mathcal{A}) - \mathcal{R}(\lambda;\mathcal{A})}{\xi - \lambda} z_0 \, d\xi \\
&= -\frac{1}{2\pi i}\int_{C_j} \frac{\mathcal{R}(\xi;\mathcal{A})}{\xi - \lambda} z_0 \, d\xi + \left(\frac{1}{2\pi i}\int_{C_j} \frac{1}{\xi - \lambda}\, d\xi\right) \mathcal{R}(\lambda;\mathcal{A})\, z_0.
\end{aligned}$$

Since λ lies outside C_j, the second integrand is analytic in some disk enclosing C_j and thus, by Cauchy's theorem, the second integral is zero. This gives

$$\mathcal{R}(\lambda;\mathcal{A})\,\mathcal{P}_{\mu_j} z_0 = -\frac{1}{2\pi i}\int_{C_j} \frac{\mathcal{R}(\xi;\mathcal{A})}{\xi - \lambda} z_0 \, d\xi.$$

Once again this integral can be taken in $\mathcal{D}(\mathcal{A})$. Thus, applying $\mathcal{C}$ we have

$$\mathcal{C}\mathcal{R}(\lambda;\mathcal{A})\,\mathcal{P}_{\mu_j} z_0 = -\frac{1}{2\pi i}\int_{C_j} \frac{\mathcal{C}\mathcal{R}(\xi;\mathcal{A})}{\xi - \lambda} z_0 \, d\xi.$$

Using the assumption we obtain $\mathcal{C}\mathcal{R}(\lambda;\mathcal{A})\,\mathcal{P}_{\mu_j} z_0 = 0$ for every such λ, and thus, by analytic continuation, for every $\lambda \in \rho(\mathcal{A})$. □

We are now ready to prove Theorem 1.1. Let us just introduce a last definition for commodity. For a subspace $E \subset H$ invariant under $\mathcal{S}(t)$, we say that the pair $(\mathcal{A}, \mathcal{C})$ is observable in E if

$$\forall z_0 \in E, \quad \Big(\mathcal{C}\mathcal{S}(t) z_0 = 0 \text{ for a.e. } t \in (0, +\infty)\Big) \Longrightarrow z_0 = 0.$$

Proof of Theorem 1.1. We will prove that the following properties are equivalent:

1. The pair $(\mathcal{A}, \mathcal{C})$ is observable in every eigenspace of $\mathcal{A}$.
2. The pair $(\mathcal{A}, \mathcal{C})$ is observable in every root subspace of $\mathcal{A}$.
3. The pair $(\mathcal{A}, \mathcal{C})$ is observable in $\mathcal{D}(\mathcal{A})$.

It is adapted from Corollary 3.2 and Corollary 3.3 of [12]. We recall that the first condition is equivalent to: $\operatorname{Ker}(s - \mathcal{A}) \cap \operatorname{Ker}(\mathcal{C}) = \{0\}$ for every $s \in \mathbb{C}$ (see Remark 1.2).

The scheme of the proof is 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longrightarrow$ 1. (the last implication is obvious).

Assume that the pair $(\mathcal{A},\mathcal{C})$ is observable in every eigenspace. If z_0 belongs to the root subspace of $\mathcal{A}$ associated with μ_j, then $\mathcal{S}(t)z_0$ is a polynomial in t, up to a factor $e^{\mu_j t}$:

$$\mathcal{S}(t)z_0 = e^{\mu_j t}p_j(t),$$

with

$$p_j(t) = \sum_{\sigma=0}^{\tau_j-1} a_{j,\sigma}t^\sigma, \quad a_{j,\sigma} = \frac{(-1)^\sigma}{\sigma!}(\mu_j - \mathcal{A})^\sigma z_0.$$

This can be seen using the uniqueness of the solution to the evolution equation satisfied by $\mathcal{S}(\cdot)z_0$. Thus, the identity $\mathcal{C}\mathcal{S}(\cdot)z_0 = 0$ reads

$$\mathcal{C}(\mu_j - \mathcal{A})^\sigma z_0 = 0, \quad 0 \leq \sigma \leq \tau_j - 1.$$

In particular for $\sigma = \tau_j - 1$ we have

$$\mathcal{C}(\mu_j - \mathcal{A})^{\tau_j-1}z_0 = 0.$$

Now, recall that z_0 lies in the root subspace $\operatorname{Ker}(\mu_j - \mathcal{A})^{\tau_j}$, so that

$$(\mu_j - \mathcal{A})^{\tau_j-1}z_0 \in \operatorname{Ker}(\mu_j - \mathcal{A}).$$

Thus, the assumption implies that

$$(\mu_j - \mathcal{A})^{\tau_j-1}z_0 = 0. \tag{20}$$

Taking this time $\sigma = \tau_j - 2$ we have

$$\mathcal{C}(\mu_j - \mathcal{A})^{\tau_j-2}z_0 = 0,$$

and from (20) we know that

$$(\mu_j - \mathcal{A})^{\tau_j-2}z_0 \in \operatorname{Ker}(\mu_j - \mathcal{A}),$$

so that the assumption gives

$$(\mu_j - \mathcal{A})^{\tau_j-2}z_0 = 0.$$

Iterating this process we obtain $z_0 = 0$.

Assume now that the pair $(\mathcal{A},\mathcal{C})$ is observable in every root subspace and let $z_0 \in \mathcal{D}(\mathcal{A})$ be such that $\mathcal{C}\mathcal{S}(t)z_0 = 0$. Applying Lemma A.1 and Lemma A.2 we obtain that $\mathcal{C}\mathcal{S}(t)\mathcal{P}_{\mu_j}z_0 = 0$ for a.e. $t \in (0,+\infty)$ and every j. By assumption we deduce that $\mathcal{P}_{\mu_j}z_0 = 0$ for every j, that is, $z_0 \in \left(\operatorname{Im}\left(\mathcal{P}^*_{\mu_j}\right)\right)^\perp$ for every j. Since the system of root vectors of $\mathcal{A}^*$ is assumed to be complete in H, we conclude that $z_0 = 0$. □

Acknowledgments. The author would like to thank Assia Benabdallah and Franck Boyer for their advices and constant support. The author also thanks Mehdi Badra and Takéo Takahashi for mentioning the paper [12].

REFERENCES

[1] F. Alabau-Boussouira and M. Léautaud, Indirect controllability of locally coupled wave-type systems and applications, *J. Math. Pures Appl.,* **99** (2013), 544–576.

[2] F. Ammar-Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A Kalman rank condition for the localized distributed controllability of a class of linear parbolic systems, *J. Evol. Equ.,* **9** (2009), 267–291.

[3] F. Ammar-Khodja, A. Benabdallah, C. Dupaix and M. González-Burgos, A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems, *Differ. Equ. Appl.,* **1** (2009), 427–457.

[4] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials, *J. Math. Pures Appl.,* **96** (2011), 555–590.

[5] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, Recent results on the controllability of linear coupled parabolic problems: A survey, *Math. Control Relat. Fields,* **1** (2011), 267–306.

[6] M. Badra and T. Takahashi, On the Fattorini criterion for approximate controllability and stabilizability of parabolic systems, to appear in *ESAIM Control Optim. Calc. Var.,* (2014). Available from: http://hal.archives-ouvertes.fr/hal-00743899.

[7] A. Benabdallah, F. Boyer, M. González-Burgos and G. Olive, Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null-controllability in cylindrical domains, to appear in *SIAM J. Control Optim.,* (2014). Available from: http://hal.archives-ouvertes.fr/hal-00845994.

[8] A. Benabdallah, M. Cristofol, P. Gaitan and L. de Teresa, A new Carleman inequality for parabolic systems with a single observation and applications, *C. R. Math. Acad. Sci. Paris,* **348** (2010), 25–29.

[9] F. Boyer and G. Olive, Approximate controllability conditions for some linear 1D parabolic systems with space-dependent coefficients, to appear in *Math. Control Relat. Fields,* (2014). Available from: http://hal.archives-ouvertes.fr/hal-00848709.

[10] N. Dunford and J. T. Schwartz, *Linear Operators. Part III : Spectral Operators,* Wiley-Interscience, New York, 1971.

[11] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations,* Springer, New York, 2000.

[12] H. O. Fattorini, Some remarks on complete controllability, *SIAM J. Control,* **4** (1966), 686–694.

[13] H. O. Fattorini, Boundary control of temperature distributions in a parallelepipedon, *SIAM J. Control,* **13** (1975), 1–13.

[14] E. Fernández-Cara, M. González-Burgos and L. de Teresa, Boundary controllability of parabolic coupled equations, *J. Funct.Anal.,* **259** (2010), 1720–1758.

[15] M. González-Burgos and L. de Teresa, Controllability results for cascade systems of m coupled parabolic PDEs by one control force, *Port. Math.,* **67** (2010), 91–113.

[16] S. Guerrero, Null controllability of some systems of two parabolic equations with one control force, *SIAM J. Control Optim.,* **46** (2007), 379–394.

[17] M. L. J. Hautus, Controllability and observability conditions for linear autonomous systems, *Ned. Akad. Wetenschappen, Proc. Ser. A,* **31** (1969), 443–448.

[18] L. Hörmander, *Linear Partial Differential Operators,* Springer Verlag, Berlin-New York, 1976.

[19] O. Kavian and L. de Teresa, Unique continuation principle for systems of parabolic equations, *ESAIM Control Optim. Calc. Var.,* **16** (2010), 247–274.

[20] R. C. MacCamy, V. J. Mizel and T. I. Seidman, Approximate boundary controllability for the heat equation, *Jour. of Math. Anal. and Appl.,* **23** (1968), 699–703.

[21] A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils* Amer. Math. Soc. 71, Providence (R.I.), 1988.

[22] K. Mauffrey, On the null controllability of a 3×3 parabolic system with non-constant coefficients by one or two control forces, *J. Math. Pures Appl.,* **99** (2013), 187–210.

[23] L. Miller, On the null-controllability of the heat equation in unbounded domains, *Bull. Sci. Math.,* **129** (2005), 175–185.

[24] G. Olive, Null-controllability for some linear parabolic systems with controls acting on different parts of the domain and its boundary, *Math. Control Signals Systems,* **23** (2012), 257–280.

[25] L. Rosier and L. de Teresa, Exact controllability of a cascade system of conservative equations, *C. R. Math. Acad. Sci. Paris,* **349** (2011), 291–296.

[26] L. de Teresa, Insensitizing controls for a semilinear heat equation, *Comm. Partial Differential Equations,* **25** (2000), 39–72.

Received April 2013; revised December 2013.

E-mail address: guillaume.olive@univ-amu.fr