Stabilization and controllability of first-order integro-differential hyperbolic equations

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joint work with

Jean-Michel CORON and Long HU

Nonlinear Partial Differential Equations and Applications

– A conference in the honor of Jean-Michel CORON for his 60th birthday –

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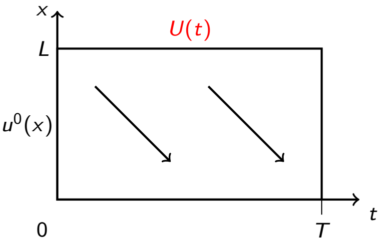
We consider

$$\begin{cases} u_t(t,x) - u_x(t,x) = \displaystyle\int_0^L g(x,y)u(t,y)\,dy & t \in (0,T), \\ u(t,L) = U(t) & x \in (0,L), \\ u(0,x) = u^0(x), & \end{cases} \quad (1)$$

where :

- $T > 0$ is the time of control and $L > 0$ is the length of the domain.
- u^0 is the initial data and u is the state.
- $g \in L^2((0,L)\times(0,L))$ is a given kernel.
- $U \in L^2(0,T)$ is a boundary control.



An application : PDE-ODE systems

Example borrowed from A. Smyshlyaev and M. Krstic (2008) :

$$\begin{cases} u_t(t,x) - u_x(t,x) &= v(t,x), \\ u(t,L) &= U(t), \\ u(0,x) &= u^0(x), \end{cases} \qquad \begin{cases} v_{xx}(t,x) - v(t,x) &= u(t,x), \\ v_x(t,0) &= 0, \\ v(t,L) &= V(t). \end{cases} \qquad \begin{matrix} t \in (0,T), \\ x \in (0,L). \end{matrix}$$

Can we find U, V as functions of u, v such that, for some $T > 0$,

$$u(T,\cdot) = v(T,\cdot) = 0 \quad ?$$

(remark : $u(T,\cdot) = 0 \Longrightarrow v(T,\cdot) = 0$).

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$$v(t,x) = \frac{\cosh(x)}{\cosh(L)} \Bigg(V(t) - \underbrace{\int_0^L u(t,y) \sinh(L-y)\, dy}_{\text{Fredholm}} \Bigg) + \underbrace{\int_0^x u(t,y) \sinh(x-y)\, dy}_{\text{Volterra}}.$$

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- If we have 2 controls : take V such that $v(t,0) = 0$: Volterra integral.
- If we have 1 control ($V = 0$) : Fredholm integral.

Abstract form of (1)

Let us rewrite (1) in the abstract form in $L^2(0, L)$:

$$\begin{cases} \dfrac{d}{dt}u = & Au + BU, \quad t \in (0, T), \\ u(0) = & u^0, \end{cases}$$

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$$Au = u_x + \int_0^L g(\cdot, y)u(y)\,dy,$$

with domain $D(A) = \left\{ u \in H^1(0, L) \,\middle|\, u(L) = 0 \right\}$, and $B \in \mathcal{L}(\mathbb{C}, D(A^*)')$ is

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We can show that, there exists a unique solution (by transposition)

$$u \in C^0([0, T]; L^2(0, L)).$$

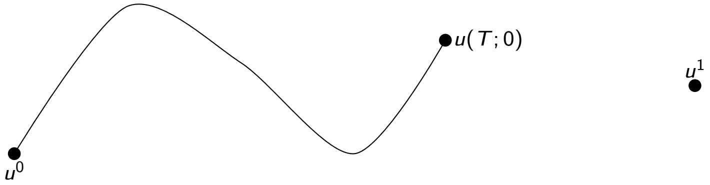

FIGURE – Uncontrolled trajectory

- u^0 : initial state, u^1 : target,
- $u(T;U)$: value of the solution to (1) at time T with control U.

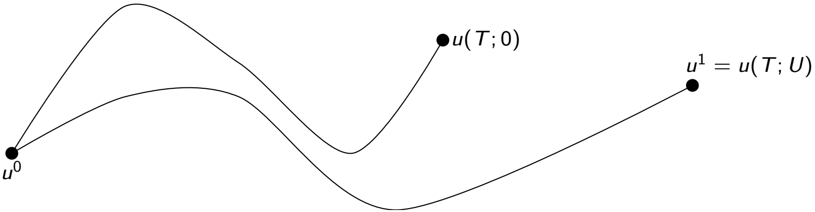

FIGURE – Trajectory controlled exactly

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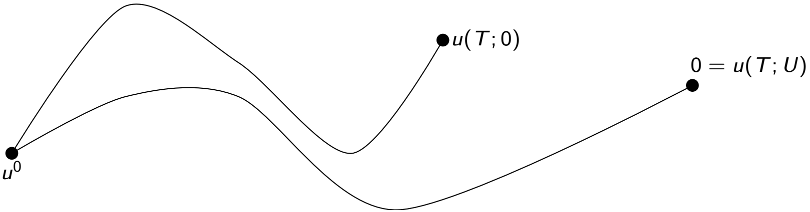

FIGURE – Trajectory controlled to 0

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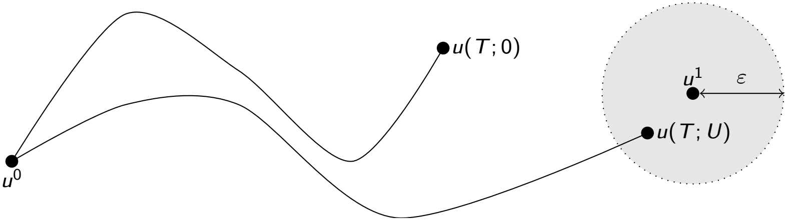

FIGURE – Trajectory controlled approximatively

- u^0 : initial state, u^1 : target,
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Stability ($U(t) = 0$) : We say that (1) is

- exp. stable if the solution u with $U(t) = 0$ satisfies

$$\|u(t)\|_{L^2} \leq M_\omega e^{-\omega t}, \quad \forall t \geq 0, \tag{2}$$

  for some $\omega > 0$ and $M_\omega > 0$.

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References on the stabilization of (1)

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- (1) is stabilizable in finite time $T = L$, if
  - g is small enough.

  or

  - $g(x,y) = g_2(y)$ with $1 - \int_0^L g_2(y) \left(\int_y^L e^{-\lambda_0 (x-y)} \, dx \right) dy \neq 0$, where $\lambda_0 = \int_0^L g_2(y) \, dy$.

F. Argomedo-Bribiesca and M. Krstic (2015)

Theorem

Assume that :

$$\begin{cases} \text{There exists } \theta \in H^1(\mathcal{T}_+) \cap H^1(\mathcal{T}_-) \text{ such that (a.e.) :} \\ \theta_x(x,y) + \theta_y(x,y) + \displaystyle\int_0^L \overline{g(y,\sigma)}\theta(x,\sigma)d\sigma = \overline{g(y,x)}, \qquad x,y \in (0,L). \\ \theta(0,y) = 0, \quad \theta(L,y) = 0, \end{cases} \tag{E}$$

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  Known as "Hautus test" although the work of Hautus (1969) is posterior to the work of Fattorini (1966).
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- (Fatt) can fail for an arbitrary large number of λ.
- Important corollary : all the notions of controllability/stabilizability are equivalent, under assumption (E).

Find F and P such that

$$\underbrace{\begin{cases} \frac{d}{dt}u = & Au + B\left(Fu\right), \\ u(0) = & u^0. \end{cases}}_{\text{(initial system)}} \xleftarrow{\text{transformation } P} \underbrace{\begin{cases} \frac{d}{dt}w = & A_0 w, \\ w(0) = & w^0. \end{cases}}_{\text{(target system)}}$$

where :

- The target system is stable.
- P is invertible.

Formally, (P, F) has to satisfy

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Basic idea of Backstepping and finite dimension

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In the finite dimensional case, taking $A_0 = A - \lambda$ with $\lambda > 0$ large enough, we have

Theorem (J.-M. Coron (2015))

If (A, B) is controllable, then there exists a solution (P, F) to (4). Moreover, this solution is unique if

$$PB = B.$$

Choice of target system

For equation (1), we choose as target system

$$\begin{cases} w_t(t,x) - w_x(t,x) = & 0, \\ w(t,L) = & 0, \\ w(0,x) = & w^0(x), \end{cases} \qquad t \in (0,+\infty),\ x \in (0,L), \tag{5}$$

which is stable in finite time $T = L$:

$$w(t,\cdot) = 0, \quad \forall t \geq L.$$

Choice of the transformation

We look for $P : L^2(0, L) \longrightarrow L^2(0, L)$ in the form

$$P = \mathrm{Id} - K,$$

where, additionally, K is an integral operator with kernel k :

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This kind of transformation (Fredholm) has already been used in :

- J.-M. Coron and Q. Lü (2014) for the rap. stabilization of a Korteweg-de Vries equation.
- J.-M. Coron and Q. Lü (2015) for the rap. stabilization of a Kuramoto-Sivashinsky equ.
- F. Argomedo-Bribiesca and M. Krstic (2015) for (1).

Formal derivation of the kernel equation

Derivating (Fred-transfo) w.r.t t gives

$$\begin{aligned} u_t(t,x) =& w_t(t,x) - \int_0^L k(x,y) w_t(t,y) dy \\ =& w_x(t,x) - \int_0^L k(x,y) w_y(t,y)\, dy \\ =& w_x(t,x) + \int_0^L k_y(x,y) w(t,y)\, dy - \cancel{k(x,L) \underbrace{w(t,L)}_{=0}} + k(x,0) w(t,0). \end{aligned}$$

Formal derivation of the kernel equation

Derivating (Fred-transfo) w.r.t t gives

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Derivating (Fred-transfo) w.r.t x gives

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As a result, k has to satisfy the following kernel equation :

$$\begin{cases} k_y(x,y) + k_x(x,y) + \displaystyle\int_0^L g(x,\sigma) k(\sigma,y) d\sigma = g(x,y), \\ k(x,0) = 0. \end{cases}$$

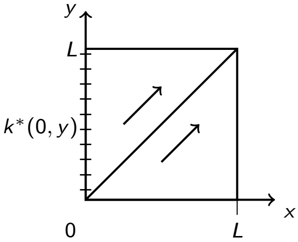
The equation of the adjoint kernel

Let us introduce the adjoint kernel

$$k^*(x,y) = \overline{k(y,x)}.$$

Then, k^* has to verify

$$\begin{cases} k^*_x(x,y) + k^*_y(x,y) + \displaystyle\int_0^L \overline{g(y,\sigma)} k^*(x,\sigma) d\sigma = \overline{g(y,x)}, \\ k^*(0,y) = 0, \end{cases} \qquad x, y \in (0,L).$$



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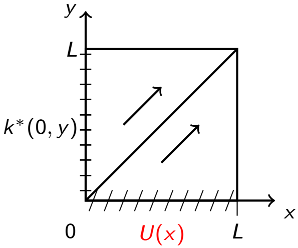

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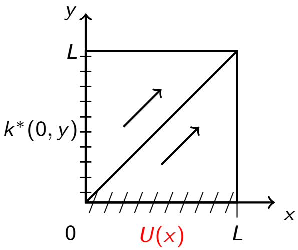

PROBLEM : not every corresponding (Fred-transfo) is invertible.

With the assumption (E), we assume that there exists U such that the solution to

$$\begin{cases} k^*_x(x,y) + k^*_y(x,y) + \displaystyle\int_0^L \overline{g(y,\sigma)} k^*(x,\sigma) d\sigma = \overline{g(y,x)}, \\ k^*(x,0) = U(x), \\ k^*(0,y) = 0, \end{cases} \qquad x, y \in (0, L),$$

satisfies the final condition

$$k^*(L,\cdot) = 0.$$

We will prove that (Fred-transfo) is then invertible, if (Fatt) holds.

Invertibility of the transformation

We want to prove that $P = \mathrm{Id} - K$ is invertible. Clearly,

$$\mathrm{Id} - K \text{ is invertible} \quad \Longleftrightarrow \quad \mathrm{Id} - K^* \text{ is invertible.}$$

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Since N is finite dimensional, ${A^*}_{|N}$ has at least one eigenfunction : $A^*\xi = \lambda\xi$, $\xi \in N$, $\xi \neq 0$. Thus,

$$\xi \in \ker(\lambda - A^*) \cap \ker B^*,$$

but

$$\xi \neq 0,$$

a contradiction with (Fatt). □

$u = Pw$ versus $w = Qu$.

Let us denote $Q = P^{-1}$. Then Q is also a Fredholm transformation :

$$Q = \mathrm{Id} - H,$$

with

$$Hu(x) = \int_0^L h(x,y)u(y)\,dy.$$

Moreover, the kernel h satisfies

$$\begin{cases} h_x(x,y) + h_y(x,y) - \displaystyle\int_0^L g(\sigma,y)h(x,\sigma)d\sigma = -g(x,y), \\ h(x,0) = 0, \quad h(x,L) = 0, \end{cases} \qquad x,y \in (0,L).$$

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REMARK : this transformation is always invertible.
Deduction : the existence should be more difficult (one has to use (Fatt) at some point).

An example : separated variables

Let us treat the case

$$g(x,y) = g_1(x)g_2(y).$$

In this case, there exists a solution to

$$\begin{cases} \theta_x(x,y) + \theta_y(x,y) + \displaystyle\int_0^L \overline{g(y,\sigma)}\theta(x,\sigma)d\sigma = \overline{g(y,x)}, & x,y \in (0,L), \\ \theta(0,y) = 0, \quad \theta(L,y) = 0, & \end{cases}$$

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and that (Fatt) is equivalent to

$$\int_0^L e^{-\lambda x}\overline{g_1(x)}\left(\int_0^x e^{\lambda y}\overline{g_2(y)}\,dy\right)dx\neq 1, \quad \forall\lambda\in Z(g_2),$$

where $Z(g_2)=\left\{\lambda\in\mathbb{C}\ :\ \int_0^L e^{\lambda y}\overline{g_2(y)}\,dy=0\right\}$.

An example : separated variables

If we assume

$$g(x,y) = g_1(x),$$

then (Fatt) is equivalent to

$$\frac{1}{\lambda_k}\left(\lambda_0 - \int_0^L e^{-\lambda_k x}\overline{g_1(x)}\,dx\right) \neq 1, \quad \forall k \neq 0 \quad (k \in \mathbb{Z}), \tag{6}$$

where $\lambda_k = \frac{2k\pi}{L} i$ for $k \neq 0$ and $\lambda_0 = \int_0^L \overline{g_1(x)}\,dx$.

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On the other hand, (6) can fail for an arbitrary large number N of k. For instance :

$$g(x,y) = g_1(x) = \frac{2}{L}\sum_{k=1}^{N} \frac{2k\pi}{L} \sin\left(\frac{2k\pi}{L}x\right).$$

Finally, if

$$g(x,y) = g_2(y),$$

then (Fatt) is equivalent to

$$\begin{cases} \displaystyle\int_0^L e^{\lambda_0 y}\overline{g_2(y)}\,dy \neq 0 & \text{si } \lambda_0 \neq 0, \\ \displaystyle -\int_0^L y\,\overline{g_2(y)}\,dy \neq 1 & \text{si } \lambda_0 = 0, \end{cases}$$

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Equivalent to the condition of F. Argomedo-Bribiesca and M. Krstic (2015)
The kernels are different :

$$\theta(x,y) = \begin{cases} \displaystyle\int_0^x \overline{g_2(y)}\,dy, & \text{si } (x,y) \in \mathcal{T}_+, \\ \displaystyle -\int_x^L \overline{g_2(y)}\,dy, & \text{si } (x,y) \in \mathcal{T}_-, \end{cases} \quad \neq \quad \theta(x,y) = \int_0^x e^{-\lambda_0(x-y)}\overline{g_2(y)}\,dy,$$

(unless $\lambda_0 = 0$).

Open problems

- Does the kernel equation (with zero final condition) always possess a solution ? Is it unique if (Fatt) ?
- Is a L^2-regularity of the kernel enough to ensure the invertibility ?
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Thank you for your attention !