# Introduction to linear control theory

Lecture notes, Shandong University

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# Contents

# Chapter 1

# Controllability of time-invariant linear O.D.E.s

In all this chapter, $\cdot$ stands for any inner product in $\mathbb{R}^n$ (or $\mathbb{R}^m$) and $\|\cdot\|$ denotes the associated norm. Since all the inner products are equivalent in finite dimension, what follows does not depend on the choice of this particular inner product.

## 1.1 Introduction

In this chapter we focus on the $n \times n$ time-invariant linear O.D.E.

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}y & = & Ay + Bu, \quad t \in (0,T), \\ y(0) & = & y^0, \end{array} \right. \tag{1.1}$$

where:

- $T > 0$ is a given time called time of control,
- $y^0 = (y_1^0, \dots, y_n^0)$ is the initial data,
- $y = (y_1, \dots, y_n)$ is the state,
- $A \in \mathbb{R}^{n\times n}$ is a matrix that couples the equations of the system,
- $u = (u_1, \dots, u_m)$ are at our disposal, they are the so-called controls,
- $B \in \mathbb{R}^{n\times m}$ is a matrix that localizes the controls on the equations.

We recall that (1.1) is well-posed: for every $y^0 \in \mathbb{R}^n$ and every $u \in L^2(0,T)^m$, there exists a unique solution $y \in H^1(0,T)^n$ to the system (1.1) given by the Duhamel's formula

$$y(t) = e^{tA}y^0 + \int_0^t e^{(t-s)A}Bu(s)\,ds, \quad \forall t \in [0,T]. \tag{1.2}$$

Note in particular that

$$y \in C^0([0,T])^n,$$

which is crucial to define the different notions of controllability. Finally, note that

$$\|y(t)\| \leq C\left(\left\|y^0\right\| + \|u\|_{L^2(0,T)^m}\right), \quad \forall t \in [0,T], \tag{1.3}$$

for some $C > 0$ that does not depend on $y^0$ nor on $u$.

**Definition 1.1.1** (Controllability)**.** We say that the system (1.1) is:

(i) exactly controllable in time $T$ if, for every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to system (1.1) satisfies

$$y(T) = y^1.$$

(ii) null-controllable in time $T$ if the above property holds for $y^1 = 0$.

(iii) approximately controllable in time $T$ if, for every $\varepsilon > 0$ and every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to system (1.1) satisfies

$$\left\|y(T) - y^1\right\| \leq \varepsilon.$$

**Example 1.1.2.** *If $m = n$ and $B = \mathrm{Id}$, then (1.1) is exactly controllable in time $T$ for every $T > 0$. Indeed, it is enough to take any smooth function $y$ with $y(0) = y^0$ and $y(T) = y^1$ and set $u = \frac{d}{dt}y - Ay$.*

*Remark* 1.1.3. Clearly, exact controllability in time $T$ implies null and approximate controllability in the same time $T$.

*Remark* 1.1.4. Let us consider the nonhomogeneous system

$$\begin{cases} \dfrac{d}{dt}y = Ay + Bu + f(t), & t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{1.4}$$

where $f \in L^2(0,T)^n$. Then, we can define the corresponding notions of controllability exactly as in Definition 1.1.1, where instead $y$ is now the solution to (1.4). It turns out that, if (1.1) is exactly controllable in time $T$, then (1.4) is exactly controllable in time $T$ for every $f \in L^2(0,T)^n$ (the converse being obvious, we see that it is enough to only study the exact controllability of (1.1)). Indeed, firstly we consider the nonhomogeneous free system (that is without controls):

$$\begin{cases} \dfrac{d}{dt}\bar{y} = A\bar{y} + f(t), & t \in (0,T), \\ \bar{y}(0) = y^0, \end{cases}$$

and then we take a control that steers in time $T$ the solution to (1.1) from 0 to $y^1 - \bar{y}(T)$.

Let us now reformulate the different notions of controllability. To this goal we introduce the linear operators

$$\begin{array}{rccc} F_T & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & y^0 & \longmapsto & \overline{y}(T), \end{array}$$

where $\overline{y}$ is the solution to the free system:

$$\begin{cases} \dfrac{d}{dt}\overline{y} & = & A\overline{y}, \quad t \in (0,T), \\ \overline{y}(0) & = & y^0, \end{cases}$$

and

$$\begin{array}{rccc} G_T & : & L^2(0,T)^m & \longrightarrow & \mathbb{R}^n \\ & & u & \longmapsto & \widehat{y}(T), \end{array} \tag{1.5}$$

where $\hat{y}$ is the solution to the nonhomogeneous system with zero initial data:

$$\begin{cases} \dfrac{d}{dt}\widehat{y} & = & A\widehat{y} + Bu, \quad t \in (0,T), \\ \widehat{y}(0) & = & 0. \end{cases}$$

With these notations, we have

$$\begin{aligned} y(T) &= \overline{y}(T) + \hat{y}(T) \\ &= F_T y^0 + G_T u, \end{aligned} \tag{1.6}$$

where $y$ is the solution to (1.1). It follows that:

(i) (1.1) is exactly controllable in time $T$ if, and only if,

$$\operatorname{Im} G_T = \mathbb{R}^n. \tag{1.7}$$

(ii) (1.1) is null-controllable in time $T$ if, and only if,

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T. \tag{1.8}$$

(iii) (1.1) is approximately controllable in time $T$ if, and only if,

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n, \tag{1.9}$$

where $\overline{\operatorname{Im} G_T}$ denotes the closure of the set $\operatorname{Im} G_T$.

As a consequence of these reformulations we see that all the notions of controllability are equivalent for the finite dimensional system (1.1):

**Proposition 1.1.5.** *Let $T > 0$. The following statements are equivalent:*

*(i)* (1.1) *is exactly controllable in time* $T$.

*(ii)* (1.1) *is null-controllable in time* $T$.

*(iii)* (1.1) *is approximately controllable in time* $T$.

*Therefore, from now on, we shall only say "controllable in time* $T$*".*

*Proof.* Since $\operatorname{Im} F_T = \mathbb{R}^n$, it is clear that (1.7) and (1.8) are equivalent. On the other hand, (1.7) and (1.9) are clearly equivalent since $\operatorname{Im} G_T$ is a finite dimensional subspace and therefore it is closed. □

*Remark* 1.1.6. We arbitrarily chose to consider controls which are in $L^2(0,T)^m$ but let us mention that we can actually consider any dense subspace of $L^2(0,T)^m$ as control set. Indeed, for any subspace $V \subset L^2(0,T)^m$, we have

$$\operatorname{Im} G_{T|\overline{V}} \subset \overline{\operatorname{Im} G_{T|V}} = \operatorname{Im} G_{T|V},$$

where the inclusion holds because $G_T$ is continuous (see (1.3)) and the equality holds because $\operatorname{Im} G_{T|V}$ is finite dimensional. Thus, if $\overline{V} = L^2(0,T)^m$ and $\operatorname{Im} G_T = \mathbb{R}^n$, then $\operatorname{Im} G_{T|V} = \mathbb{R}^n$. In particular, if there exists a control which is barely in $L^2(0,T)^m$, then there exists as well a control which is smooth, say in $C_c^\infty(0,T)^m$.

## 1.2 Duality

The duality is based on the following general result:

**Lemma 1.2.1.** *Let* $H_1, H_2$ *be two Hilbert spaces and* $G \in \mathcal{L}(H_1, H_2)$ *be a bounded linear operator. Then,*

$$\overline{\operatorname{Im} G} = (\ker G^*)^\perp.$$

Thanks to (1.3) we see that $G_T \in \mathcal{L}(L^2(0,T)^m, \mathbb{R}^n)$. Thus, by Lemma 1.2.1, we have:

**Proposition 1.2.2.** (1.1) *is controllable in time* $T$ *if, and only if,*

$$\ker G_T^* = \{0\}.$$

Let us now compute $G_T^*$. Note that this can easily be done using the definition (1.5) of $G_T$ and the explicit formula (1.2) but let us provide a different proof. To this end, we take the inner product of the equation (1.1) with a smooth function $z$ and we integrate by parts to obtain the relation

$$y(T) \cdot z(T) - y(0) \cdot z(0) + \int_0^T y(t) \cdot \left( -\frac{d}{dt} z(t) - A^* z(t) \right) dt = \int_0^T u(t) \cdot B^* z(t) \, dt.$$

In particular, we see that, if $z$ is the solution to the following system (called adjoint system):

$$\left\{\begin{array}{rcl} -\dfrac{d}{dt}z & = & A^*z, \quad t \in (0,T), \\ z(T) & = & z^1, \end{array}\right. \tag{1.10}$$

where $z^1 \in \mathbb{R}^n$, then we have established the following fundamental relation:

$$y(T)\cdot z^1 - y^0 \cdot z(0) = \int_0^T u(t)\cdot B^* z(t)\,dt, \tag{1.11}$$

valid for every $y^0 \in \mathbb{R}^n$, $z^1 \in \mathbb{R}^n$ and $u \in L^2(0,T)^m$. Thanks to (1.11) we readily see that

$$\begin{array}{rcccc} G_T^* & : & \mathbb{R}^n & \longrightarrow & L^2(0,T)^m \\ & & z^1 & \longmapsto & B^*z. \end{array} \tag{1.12}$$

Therefore, Proposition 1.2.2 can be restated into the following fundamental result:

**Theorem 1.2.3** (Duality)**.** (1.1) *is controllable in time $T$ if, and only if,*

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^*z(t) = 0, \quad \forall t \in [0,T]\Big) \Longrightarrow z^1 = 0. \tag{1.13}$$

*Remark* 1.2.4. Clearly, (1.13) is equivalent to

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^*\tilde{z}(t) = 0, \quad \forall t \in [0,T]\Big) \Longrightarrow z^1 = 0,$$

where $\tilde{z}(t) = z(T-t)$. But $\tilde{z}$ is analytic on $(0,+\infty)$. Thus, (1.13) holds if, and only if,

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^*\tilde{z}(t) = 0, \quad \forall t \in [0,+\infty)\Big) \Longrightarrow z^1 = 0.$$

Therefore, the controllability of (1.1) does not depend on the time of control $T$. In other words, if there exists $T > 0$ such that (1.1) is controllable in time $T$, then, for every $T > 0$, (1.1) is controllable in time $T$. For this reason, in the sequel we shall only say that (1.1) is "controllable".

*Remark* 1.2.5. The strength of the duality is that it reduces the task of proving an existence result (existence of a control) to the task of proving a uniqueness result, which is often easier to handle.

## 1.3 Conditions of controllability

### 1.3.1 Gramian of controllability

**Theorem 1.3.1.** *Let $T > 0$.* (1.1) *is controllable if, and only if, the $n \times n$ matrix*

$$\Lambda_T = \int_0^T e^{(T-t)A}BB^*e^{(T-t)A^*}\,dt, \tag{1.14}$$

*is invertible. $\Lambda_T$ is called the Gramian of controllability or HUM operator.*

*Remark* 1.3.2. Note that $\Lambda_T$ is always symmetric and positive semi-definite. In particular, it is invertible if, and only if, it is positive definite. Now observe that $\Lambda_T$ is positive definite if, and only if, there exists $C > 0$ such that

$$\left\|z^1\right\|^2 \leq C^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n. \tag{1.15}$$

This inequality is called observability inequality and the best constant $C > 0$ in (1.15) is called the control cost. We shall come back to this notion later on in Section 1.4.2.

*Proof.* By Proposition 1.2.2, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Clearly, this is equivalent to

$$\ker G_T G_T^* = \{0\}.$$

By definition of $G_T$ (see (1.5)) and computation of $G_T^*$ (see (1.12)) we readily see that $G_T G_T^* = \Lambda_T$. □

### 1.3.2 Kalman rank condition

In this section we establish the following fundamental result in the control theory of finite dimensional systems:

**Theorem 1.3.3** (Kalman rank condition)**.** (1.1) *is controllable if, and only if,*

$$\operatorname{rank}\left(B|AB|\cdots|A^{n-1}B\right) = n. \tag{1.16}$$

Observe that, as expected (see Remark 1.2.4), the condition (1.16) does not depend on the time of control $T$.

The Kalman rank condition (1.16) is an easy checkable condition for the controllability.

**Example 1.3.4.** *The* $2 \times 2$ *system*

$$\begin{cases} \dfrac{d}{dt} y_1 = a_{11} y_1 + a_{12} y_2 + u, & \\ \dfrac{d}{dt} y_2 = a_{21} y_1 + a_{22} y_2, & t \in (0,T), \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, & \end{cases}$$

*is controllable if, and only if,*

$$a_{21} \neq 0.$$

*Remark* 1.3.5. Thanks to the Kalman rank condition we also see that we can fix the end-points of the control (and of its derivatives). Indeed, say that we look for controls $u$ such that, in addition,

$$u(0) = u^0, \quad u(T) = u^1,$$

for some $u^0, u^1 \in \mathbb{R}^m$. Then, to this end we consider $u$ as a new variable and we introduce the $(n+m) \times (n+m)$ augmented system

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, \\ \frac{d}{dt}u = v, \\ y(0) = y^0, \quad u(0) = u^0, \end{cases} \quad t \in (0,T),$$

where $v$ is now the control. We easily check that this system satisfies the associated Kalman rank condition.

The proof of Theorem 1.3.3 relies on the following lemma, which actually shows that $\ker G_T^*$ can be completely characterized in finite dimension:

**Lemma 1.3.6.** *For every $T > 0$, we have*

$$\ker G_T^* = \left(\mathrm{Im}\ \left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

*Proof.* By (1.12), $z^1 \in \ker G_T^*$ if, and only if,

$$B^* z(t) = 0, \quad \forall t \in [0,T], \tag{1.17}$$

where $z(t) = e^{(T-t)A^*} z^1$ is the solution to the adjoint system (1.10). Since $z$ is analytic on $(0,T)$, we have (1.17) if, and only if, for some $0 < t_0 < T$,

$$\frac{d^k}{dt^k}(B^*z)(t_0) = 0, \quad \forall k \in \{0,1,\ldots\}.$$

Computing $B^*z$ that gives

$$B^*(A^*)^k z^1 = 0, \quad \forall k \in \{0,1,\ldots\}.$$

By the Cayley-Hamilton theorem, this is equivalent to

$$B^*(A^*)^k z^1 = 0, \quad \forall k \in \{0,1,\ldots,n-1\}.$$

To summarize, $z^1 \in \ker G_T^*$ if, and only, if

$$z^1 \in \ker \begin{pmatrix} B^* \\ B^*A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix}.$$

To conclude, observe that

$$\ker \begin{pmatrix} B^* \\ B^*A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix} = \ker \left(B|AB|\cdots|A^{n-1}B\right)^* = \left(\mathrm{Im}\ \left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

□

*Proof of Theorem 1.3.3.* This is an immediate consequence of Proposition 1.2.2 and Lemma 1.3.6. □

Actually, thanks to Lemma1.3.6 we even have a stronger result than Theorem 1.3.3 which gives a precise description of the reachable states:

**Theorem 1.3.7.** *Let $y^0, y^1 \in \mathbb{R}^n$ and $T > 0$ be fixed. There exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to (1.1) satisfies $y(T) = y^1$ if, and only if,*

$$y^1 - e^{TA}y^0 \in \mathrm{Im}\ \left(B|AB|\cdots|A^{n-1}B\right).$$

*Proof.* Using (1.6) we readily see that there exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to (1.1) satisfies $y(T) = y^1$ if, and only if,

$$y^1 - e^{TA}y^0 \in \mathrm{Im}\, G_T.$$

Since $\mathrm{Im}\, G_T = (\ker G_T^*)^{\perp}$, the result follows from Lemma 1.3.6. □

There is a canonical form of controllable systems.

**Proposition 1.3.8** (Canonical form of Brunovski)**.** *Let $m = 1$. Assume that* (1.16) *holds and let $K = (B|AB|\cdots|A^{n-1}B)$ (note that $K \in \mathbb{R}^{n\times n}$). Then,*

$$K^{-1}AK = \tilde{A}, \quad K^{-1}B = \tilde{B},$$

*with*

$$\tilde{A} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \alpha_0 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{n-1} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \tag{1.18}$$

*where $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$ are the coefficients of the characteristic polynomial of $A$, namely $p(\lambda) = \det(\lambda \mathrm{Id} - A) = \lambda^n - \alpha_{n-1}\lambda^{n-1} - \ldots - \alpha_0$, $\lambda \in \mathbb{C}$ ($\tilde{A}$ is the companion matrix of $p$).*

*Proof.* The proof is a simple computation of $K\tilde{A}$ and $K\tilde{B}$. □

It is worth mentioning that, once we know the "good" condition for the controllability (namely, (1.16)), there exists a direct proof of Theorem 1.3.3. By direct proof we mean a proof that is not using the duality at all. It is based on Proposition 1.3.8 and the following result, that we shall prove in a self-contained way (see [Boy17, Chapter II, Section 2] and [Cor07, pp.13-15]):

**Proposition 1.3.9.** *Let $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$. The system*

$$\begin{cases} \dfrac{d}{dt}y &= \tilde{A}y + \tilde{B}u, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1.19}$$

*where $\tilde{A}$ and $\tilde{B}$ are given by* (1.18)*, is controllable.*

*Remark* 1.3.10. Combining Proposition 1.3.8 with Proposition 1.3.9 this gives a direct proof of the implication "$\Longleftarrow$" in Theorem 1.3.3 for $m = 1$.

*Proof of Proposition 1.3.9 (without using Theorem 1.3.3).* Firstly, we recall that it is sufficient to only consider the target $y^1 = 0$ (see Proposition 1.1.5). Let $\overline{y}$ be the free solution to (1.19), that is the solution to (1.19) with $u = 0$. Let us introduce a cut-off function $\eta \in C^\infty([0,T])$ such that

$$\eta = 1 \text{ on } [0, T/3], \quad \eta = 0 \text{ on } [2T/3, T].$$

Observe that, because of the structure (1.18), the last equation of (1.19) is

$$\frac{d}{dt}y_n = y_{n-1} + \alpha_{n-1}y_n.$$

We set

$$y_n = \eta\overline{y}_n.$$

Then, we have no choice for $y_{n-1}$ but to set

$$y_{n-1} = \frac{d}{dt}y_n - \alpha_{n-1}y_n.$$

By induction, we have to set

$$y_k = \frac{d}{dt}y_{k+1} - \alpha_k y_n, \quad \forall k \in \{n-2, \dots, 1\},$$

and then

$$u = \frac{d}{dt}y_1 - \alpha_0 y_n.$$

Finally, thanks to the definition of $\eta$, note that

$$\forall k \in \{1, \ldots, n\}, \quad \begin{cases} y_k = \overline{y}_k & \text{on } [0, T/3], \\ y_k = 0 & \text{on } [2T/3, T], \end{cases}$$

so that

$$y(0) = y^0, \quad y(T) = 0.$$

□

### 1.3.3 Fattorini-Hautus test

There is another important characterization of the controllability, which is a dual version of the Kalman rank condition:

**Theorem 1.3.11** (Fattorini-Hautus test)**.** (1.1) *is controllable if, and only if,*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{1.20}$$

*Remark* 1.3.12. Theorem 1.3.11 shows in particular that the following condition is necessary for the controllability:

$$\dim \ker(\lambda - A^*) \leq m, \quad \forall \lambda \in \mathbb{C}.$$

Indeed, assume that there exists a linearly independent family $\phi_1, \ldots, \phi_{m+1}$ of $\ker(\lambda - A^*)$. Then, $B^*\phi_1, \ldots, B^*\phi_{m+1}$ is linearly dependent as $B^* \in \mathbb{R}^{m \times n}$. Thus, there exists $(\alpha_1, \ldots, \alpha_{m+1}) \neq (0, \ldots, 0)$ such that $\sum_{k=1}^{m+1} \alpha_k B^* \phi_k = 0$. Let $z^1 = \sum_{k=1}^{m+1} \alpha_k \phi_k$. Then, $B^* z^1 = 0$. But $z^1 \in \ker(\lambda - A^*)$. Therefore, (1.20) implies that $z^1 = 0$, that is $\alpha_1 = \ldots = \alpha_{m+1} = 0$, a contradiction.

*Proof.* By Proposition 1.2.2, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Assume that $\ker G_T^* = \{0\}$. Let $z^1 \in \ker(\lambda - A^*) \cap \ker B^*$. Then, $z(t) = e^{\lambda(T-t)} z^1$ and $B^* z(t) = e^{\lambda(T-t)} B^* z^1 = 0$ for every $t \in [0, T]$. Therefore, $z^1 = 0$ by assumption. Conversely, assume that $\ker G_T^* \neq \{0\}$. Let us first prove that:

(i) $\ker G_T^* \subset \ker B^*$.

(ii) $A^*(\ker G_T^*) \subset \ker G_T^*$.

Let $z^1 \in \ker G_T^*$. Then,

$$B^* z(t) = 0, \quad \forall t \in [0, T].$$

Taking $t = T$ we obtain $B^* z^1 = 0$, that is $z^1 \in \ker B^*$. On the other hand, taking the derivative we obtain

$$B^* e^{(T-t)A^*} A^* z^1 = 0, \quad \forall t \in [0, T],$$

that is $A^*z^1 \in \ker G_T^*$. Consequently, by (ii) we see the restriction of $A^*$ to $\ker G_T^*$ is a linear operator from the finite dimensional space $\ker G_T^*$ into itself and, since $\ker G_T^* \neq \{0\}$, therefore possesses at least one complex eigenvalue. Since in addition by (i) we have $\ker G_T^* \subset \ker B^*$, this shows that there exist $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{R}^n$ with $\phi \neq 0$ such that

$$A^*\phi = \lambda\phi, \quad B^*\phi = 0.$$

This proves that (1.20) fails. □

### 1.3.4 Partial controllability

Sometimes we want to control not all but only some components of the system (1.1). This leads to the notion of partial controllability (also called output controllability in the literature).

**Definition 1.3.13** (Partial controllability)**.** Let $P \in \mathbb{R}^{p\times n}$ with $p \in \mathbb{N}^*$. We say that the system (1.1) is partially controllable if, for every $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^p$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to system (1.1) satisfies

$$Py(T) = y^1.$$

One can take for instance the projection on the first $p$ components:

$$\begin{array}{cccc} P : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^p \\ & \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} & \longmapsto & \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, \end{array}$$

where $p \in \{1, \ldots, n\}$ is the number of components we would like to control.

Mimicking the procedure developed in the previous sections, we see that (1.1) is partially controllable if, and only if,

$$\operatorname{Im} PG_T = \mathbb{R}^n.$$

This is equivalent to

$$\ker G_T^* P^* = \{0\}.$$

Thanks to the expression of $G_T^*$ (see (1.12)) we see that this is also equivalent to

$$\forall z^1 \in \mathbb{R}^p, \quad \Big(B^*z(t) = 0, \quad \text{a.e. } t \in (0,T)\Big) \Longrightarrow z^1 = 0,$$

where $z$ is the solution to the following adjoint system:

$$\begin{cases} -\dfrac{d}{dt}z &= A^*z, \quad t \in (0,T), \\ z(T) &= P^*z^1. \end{cases}$$

Reproducing the proof of Lemma 1.3.6 we easily obtain the following result:

**THEOREM 1.3.14** (Kalman rank condition)**.** *Let $P \in \mathbb{R}^{p\times n}$ with $p \in \mathbb{N}^*$.* (1.1) *is partially controllable if, and only if,*

$$\operatorname{rank}(PB|PAB|\cdots|PA^{n-1}B) = p.$$

### 1.3.5 Higher order O.D.E.s

An interesting consequence of Theorem 1.3.3 is that it also gives a characterization of the controllability of some higher order systems.

Let $y^0, \dot{y}^0 \in \mathbb{R}^n$ and let us consider the second order system:

$$\begin{cases} \dfrac{d^2}{dt^2}y = Ay + Bu, \quad t \in (0,T), \\ y(0) = y^0, \quad \dfrac{d}{dt}y(0) = \dot{y}^0. \end{cases} \tag{1.21}$$

Firstly, we should point out that there are a priori several ways to define to controllability of (1.21). Do we want to achieve $y(T) = \frac{d}{dt}y(T) = 0$ or only $y(T) = 0$ for instance ? The first goal will be refer to as controllability and the second one as partial controllability. Of course, the notion of partial controllability for (1.21) coincides with the notion of partial controllability of Section 1.3.4 for an underlying first order system, see the proof of Theorem 1.3.16 below.

**Definition 1.3.15** (Controllability)**.**

(i) We say that the system (1.21) is controllable if, for every $y^0, \dot{y}^0 \in \mathbb{R}^n$ and $y^1, \dot{y}^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to system (1.21) satisfies

$$y(T) = y^1, \quad \frac{d}{dt}y(T) = \dot{y}^1.$$

(ii) Let $P \in \mathbb{R}^{p\times 2n}$ with $p \in \mathbb{N}^*$. We say that the system (1.21) is partially controllable if, for every $y^0, \dot{y}^0 \in \mathbb{R}^n$ and $Y^1 \in \mathbb{R}^p$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution $y$ to system (1.21) satisfies

$$P\begin{pmatrix} y(T) \\ \frac{d}{dt}y(T) \end{pmatrix} = Y^1.$$

The next result is twofold. Firstly, it shows that the same Kalman rank condition as for first order systems also characterizes the controllability of the second order systems of the form (1.21). Secondly, it shows that, surprisingly enough, it is equivalent to require $y(T) = \frac{d}{dt}y(T) = 0$ or only $y(T) = 0$ (or only $\frac{d}{dt}y(T) = 0$).

**THEOREM 1.3.16.** *The following statements are equivalent:*

*(i)* (1.21) *is controllable.*

*(ii)* (1.21) *is partially controllable for* $P \in \mathbb{R}^{n\times 2n}$ *given by*

$$P : \begin{array}{ccc} \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^n \\ \begin{pmatrix} y \\ \dot{y} \end{pmatrix} & \longmapsto & y, \end{array} \qquad \textit{or} \qquad P : \begin{array}{ccc} \mathbb{R}^{2n} & \longrightarrow & \mathbb{R}^n \\ \begin{pmatrix} y \\ \dot{y} \end{pmatrix} & \longmapsto & \dot{y}. \end{array}$$

*(iii)* $\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n$.

*Proof.* Introducing the new variable

$$\tilde{y} = \begin{pmatrix} y \\ \frac{d}{dt}y \end{pmatrix} \in \mathbb{R}^{2n},$$

and

$$\tilde{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ A & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad \tilde{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathbb{R}^{2n\times m},$$

we see that the $n\times n$ second order system (1.21) is controllable (*resp.* partially controllable) if, and only if, so is the following $2n \times 2n$ first order system:

$$\begin{cases} \dfrac{d}{dt}\tilde{y} &= \tilde{A}\tilde{y} + \tilde{B}u, \quad t \in (0,T), \\ \tilde{y}(0) &= \tilde{y}^0. \end{cases} \tag{1.22}$$

By Theorem 1.3.3 (*resp.* Theorem 1.3.14), the controllability (*resp.* partial controllability) of (1.22) is equivalent to the corresponding Kalman rank condition, that is

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2n,$$

$$(\textit{resp.} \quad \operatorname{rank}(P\tilde{B}|P\tilde{A}\tilde{B}|\cdots|P\tilde{A}^{2n-1}\tilde{B}) = n).$$

A computation shows that

$$(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = \begin{pmatrix} 0 & B & 0 & AB & \cdots & 0 & A^{n-1}B \\ B & 0 & AB & 0 & \cdots & A^{n-1}B & 0 \end{pmatrix}, \tag{1.23}$$

$$(\textit{resp.} \quad (P\tilde{B}|P\tilde{A}\tilde{B}|\cdots|P\tilde{A}^{2n-1}\tilde{B}) = (B|AB|\cdots|A^{n-1}B)).$$

Therefore,

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2\operatorname{rank}(B|AB|\cdots|A^{n-1}B),$$

$$(\textit{resp.} \quad \operatorname{rank}(P\tilde{B}|P\tilde{A}\tilde{B}|\cdots|P\tilde{A}^{2n-1}\tilde{B}) = \operatorname{rank}(B|AB|\cdots|A^{n-1}B)).$$

□

## 1.4 Optimal controls

### 1.4.1 Control of minimal $L^2$-norm

Assume that (1.1) is controllable. A priori there is no reason for a control to be unique. Let $y^0, y^1 \in \mathbb{R}^n$ and let us introduce the corresponding set of controls

$$U = \left\{ u \in L^2(0,T)^m, \quad y(T) = y^1 \right\}.$$

We consider the minimization problem

$$\min_{u \in U} \frac{1}{2} \|u\|^2_{L^2(0,T)^m}.$$

A solution of this problem will be called a control of minimal $L^2$-norm.

**Theorem 1.4.1** ($L^2$-optimal control)**.** *Assume that* (1.1) *is controllable. Then, for every* $y^0, y^1 \in \mathbb{R}^n$*, there exists a unique control of minimal* $L^2$*- norm and it is given by*

$$u_{\text{opt}}(t) = B^* e^{(T-t)A^*} \Lambda_T^{-1} \left( y^1 - e^{TA} y^0 \right), \tag{1.24}$$

*where* $\Lambda_T \in \mathbb{R}^{n \times n}$ *is the Gramian of controllability (see* (1.14)*). The control* $u_{\text{opt}}$ *is also called the HUM control.*

*Remark* 1.4.2. Note that the control $u_{\text{opt}}$ is analytic on $\mathbb{R}$.

**Lemma 1.4.3** (Hilbert projection theorem)**.** *Let* $H$ *be a Hilbert space. Let* $C \subset H$ *be a nonempty closed convex. For every* $x \in H$*, there exists a unique* $p \in C$ *such that*

$$\|x - p\|_H = \min_{y \in C} \|x - y\|_H.$$

*Moreover,* $p$ *is the unique element of* $C$ *that satisfies*

$$\langle x - p, y - p \rangle \leq 0, \quad \forall y \in C.$$

*Proof of Theorem 1.4.1.* Firstly, observe that $U$ is not empty by assumption. Let then $u_0 \in U$. We easily see that

$$U = u_0 + \ker G_T.$$

Therefore, $U$ is an affine subspace of $\mathbb{R}^n$. In particular, it is a closed convex and, by Lemma 1.4.3, there exists a unique $u_{\text{opt}} \in U$ (the projection of 0 on $U$) such that

$$\|u_{\text{opt}}\|_{L^2(0,T)^m} = \min_{u \in U} \|u\|_{L^2(0,T)^m}.$$

Moreover, we have

$$\langle u_{\text{opt}}, u_{\text{opt}} - u \rangle_{L^2} \leq 0, \quad \forall u \in U.$$

Since $U = u_{\text{opt}} + \ker G_T$, this gives

$$\langle u_{\text{opt}}, v \rangle_{L^2} = 0, \quad \forall v \in \ker G_T.$$

Thus,

$$u_{\text{opt}} \in (\ker G_T)^{\perp} = \overline{\operatorname{Im} G_T^*}.$$

Then, there exists $(z_n^1)_n$ such that

$$G_T^* z_n^1 \xrightarrow[n \to +\infty]{} u_{\text{opt}}. \tag{1.25}$$

But $(z_n^1)_n$ converges. Indeed, since $G_T$ is a bounded operator, we have

$$G_T G_T^* z_n^1 \xrightarrow[n \to +\infty]{} G_T u_{\text{opt}}.$$

By Theorem 1.14, we know that $G_T G_T^* = \Lambda_T$ is invertible. Thus,

$$z_n^1 \xrightarrow[n \to +\infty]{} \Lambda_T^{-1} G_T u_{\text{opt}}.$$

Coming back to (1.25), we obtain that

$$u_{\text{opt}} = G_T^* \Lambda_T^{-1} G_T u_{\text{opt}}.$$

Finally, since $u_{\text{opt}} \in U$, we have

$$G_T u_{\text{opt}} = y^1 - F_T y^0.$$

Using the expressions of $G_T^*$ (see (1.12)) and $F_T$ we obtain (1.24). □

### 1.4.2 Control cost

In section we consider $y^0 = 0$. Assume that (1.1) is controllable. Then, the map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & L^2(0,T)^m \\ y^1 & \longmapsto & u_{\text{opt}}, \end{array}$$

is a bounded linear map (see for instance (1.24)). We denote by $C_T$ its norm of operator.

**Definition 1.4.4** (Control cost)**.** Assume that (1.1) is controllable. Then, the quantity

$$C_T = \sup_{\substack{y^1 \in \mathbb{R}^n \\ y^1 \neq 0}} \frac{\|u_{\text{opt}}\|_{L^2(0,T)^m}}{\|y^1\|} = \sup_{\substack{y^1 \in \mathbb{R}^n \\ \|y^1\| = 1}} \|u_{\text{opt}}\|_{L^2(0,T)^m}, \tag{1.26}$$

where $u_{\text{opt}}$ is the control of minimal $L^2$-norm steering the solution $y$ to (1.1) from $y^0 = 0$ to $y^1$ in time $T$, is called the control cost.

The following proposition gives a dual characterization for the control cost:

**Proposition 1.4.5.** *Assume that* (1.1) *is controllable. The control cost* $C_T$ *satisfies*

$$C_T = \sup_{\substack{z^1 \in \mathbb{R}^n \\ z^1 \neq 0}} \frac{\left\|z^1\right\|}{\sqrt{\int_0^T \|B^* z(t)\|^2 \, dt}} = \sup_{\substack{z^1 \in \mathbb{R}^n \\ \left\|z^1\right\| = 1}} \frac{1}{\sqrt{\int_0^T \|B^* z(t)\|^2 \, dt}}, \tag{1.27}$$

*where* $z$ *is the solution to the adjoint system* (1.10)*. In other words, the control cost* $C_T$ *is the best constant* $C > 0$ *such that the following inequality (called observability inequality) holds:*

$$\left\|z^1\right\|^2 \leq C^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n.$$

*Remark* 1.4.6. Since the closed unit ball is compact in $\mathbb{R}^n$, both supremum in (1.26) and in (1.27) are actually maximum.

*Proof.* By homogeneity the second equality in (1.27) is clear. Next, observe that (see (1.14))

$$\int_0^T \|B^* z(t)\|^2 \, dt = \Lambda_T z^1 \cdot z^1, \quad \forall z^1 \in \mathbb{R}^n,$$

and $\Lambda_T z^1 \cdot z^1 \neq 0$ for every $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ by controllability. Let $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ be fixed. Let $y^1 = z^1$ and let $u_{\text{opt}}$ be the associated optimal control. Using (1.11) and the Cauchy-Schwarz inequality we have

$$\left\|z^1\right\|^2 \leq \left( \int_0^T \|u_{\text{opt}}(t)\|^2 \, dt \right)^{\frac{1}{2}} \left( \Lambda_T z^1 \cdot z^1 \right)^{\frac{1}{2}}.$$

It follows that

$$\frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1} \leq \frac{\int_0^T \|u_{\text{opt}}(t)\|^2 \, dt}{\left\|z^1\right\|^2} = \frac{\int_0^T \|u_{\text{opt}}(t)\|^2 \, dt}{\left\|y^1\right\|^2}.$$

This shows that the supremum is finite with

$$\sup_{\substack{z^1 \in \mathbb{R}^n \\ z^1 \neq 0}} \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1} \leq C_T^2.$$

Conversely, let $y^1 \in \mathbb{R}^n$ with $y^1 \neq 0$ and let $u_{\text{opt}}$ be the associated optimal control. Set

$$z^1 = \Lambda_T^{-1} y^1.$$

Using (1.11) and the expression (1.24) of $u_{\text{opt}}$ we obtain

$$\Lambda_T z^1 \cdot z^1 = \int_0^T \|u_{\text{opt}}(t)\|^2 \, dt.$$

Thus,

$$\frac{\int_0^T \|u_{\mathrm{opt}}(t)\|^2\, dt}{\|y^1\|^2} = \frac{\int_0^T \|u_{\mathrm{opt}}(t)\|^2\, dt}{\|\Lambda_T z^1\|^2} = \frac{\Lambda_T z^1 \cdot z^1}{\|\Lambda_T z^1\|^2}.$$

But

$$\frac{\Lambda_T z^1 \cdot z^1}{\|\Lambda_T z^1\|^2} = \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1} \frac{\left|\Lambda_T z^1 \cdot z^1\right|^2}{\|\Lambda_T z^1\|^2 \, \|z^1\|^2} \leq \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1}.$$

This establishes the reversed inequality

$$C_T^2 \leq \sup_{\substack{z^1 \in \mathbb{R}^n \\ z^1 \neq 0}} \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1}.$$

□

**Proposition 1.4.7.** *Assume that* (1.1) *is controllable. The control cost* $C_T$ *satisfies:*

*(i)* $C_T \to +\infty$ *as* $T \to 0^+$.

*(ii)* $C_T$ *is decreasing.*

*Proof.* By Proposition 1.4.5 we have

$$C_T = \sup_{\substack{z^1 \in \mathbb{R}^n \\ \left\|z^1\right\| = 1}} \frac{1}{\sqrt{\int_0^T \|B^* \tilde{z}(t)\|^2\, dt}},$$

where $\tilde{z}(t) = z(T-t)$ does not depend on $T$. Then,

$$C_T \geq \frac{1}{\sqrt{\int_0^T \|B^* \tilde{z}(t)\|^2\, dt}} \xrightarrow[T \to 0^+]{} +\infty.$$

To prove the second point, we simply observe that, for every $T' \geq T$, we have

$$\int_0^{T'} \|B^* \tilde{z}(t)\|^2\, dt \geq \int_0^{T} \|B^* \tilde{z}(t)\|^2\, dt, \tag{1.28}$$

from which it immediately follows that

$$C_{T'} \leq C_T, \quad \forall T' \geq T.$$

□

*Remark* 1.4.8. Using Remark 1.4.6 we see that $C_T$ is actually strictly decreasing. Indeed, the inequality (1.28) is strict for $T' > T$ because we can not have $B^* \tilde{z}(t) = 0$ for $t \in [T, T']$ by controllability. Taking the inverse and then the maximum over all $z^1 \in \mathbb{R}^n$ with $\left\|z^1\right\| = 1$ we obtain that $C_{T'} < C_T$.

*Remark* 1.4.9. Since $C_T$ is decreasing and bounded from below by 0, we have $C_T \to \inf_{T>0} C_T$ as $T \to +\infty$. However, it is not true that $\inf_{T>0} C_T = 0$ in general. Indeed, assume for instance that $A$ has an unstable eigenvalue $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < 0$. Then, taking $z^1 = \phi$, where $\phi$ is a normalized eigenvector of $A^*$ associated with $\bar{\lambda}$, a computation gives

$$C_T^2 \geq \frac{-2\mathrm{Re}\,\lambda}{\|B^*\phi\|^2}, \quad \forall T > 0.$$

Therefore $\inf_{T>0} C_T > 0$. This feature can be explained by remarking that, on the one hand the system naturally dissipates to 0 in the direction of $\phi$ but on the other hand, the goal is to reach a state that can be different from 0. Of course, this also happens because we deal with the notion of exact controllability.

Let us conclude this section by mentioning that we can actually obtain a very precise asymptotic of the control cost as $T \to 0^+$ (the proof is admitted, see e.g. [Sei88]).

**Theorem 1.4.10** (Estimate of the control cost)**.** *Assume that* (1.1) *is controllable and let* $r \in \{0, \ldots, n-1\}$ *be the smallest exponent such that* $\mathrm{rank}\,(B|AB|\cdots|A^r B) = n$*. Then, there exists* $\gamma > 0$ *such that*

$$C_T \sim \frac{\gamma}{T^{r+\frac{1}{2}}} \qquad \text{as } T \to 0^+.$$

### 1.4.3 Variational approach

In this section we provide another approach to look at the optimal control problem. Let us go back to the fundamental identity (1.11) with $y^1 = 0$. We readily see that $y(T) = 0$ if, and only if,

$$0 = \int_0^T u(t) \cdot B^* z(t)\, dt + y^0 \cdot z(0), \quad \forall z^1 \in \mathbb{R}^n, \tag{1.29}$$

This identity can be viewed an optimality condition for the extremal points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$J(z^1) = \frac{1}{2} \int_0^T \|B^* z(t)\|^2 \, dt + y^0 \cdot z(0),$$

where $z$ is the solution to the adjoint system (1.10).

**Theorem 1.4.11.** *Assume that the system* (1.1) *is controllable. Then, for every* $y^0 \in \mathbb{R}^n$*, $J$ has a minimizer. Moreover, if* $z^1_{\mathrm{opt}}$ *is a minimizer of $J$ and* $z_{\mathrm{opt}}$ *denotes the corresponding solution to the adjoint system* (1.10)*, then, the solution $y$ to* (1.1) *corresponding to*

$$u_{\mathrm{opt}} = B^* z_{\mathrm{opt}},$$

*satisfies* $y(T) = 0$*. Moreover,* $u_{\mathrm{opt}}$ *is the unique null-control of minimal* $L^2$*-norm.*

**Lemma 1.4.12.** *Let $J:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous and convex function that is also coercive, that is*

$$J(z^1) \xrightarrow[\|z^1\|\to+\infty]{} +\infty.$$

*Then, $J$ has (at least one) minimizer.*

*Proof of Theorem 1.4.11.* Clearly, $J$ is continuous and convex on $\mathbb{R}^n$. Let us show that it is coercive. Let us introduce the function $N:\mathbb{R}^n \longrightarrow \mathbb{R}_+$ defined by

$$N(z^1) = \int_0^T \|B^* z(t)\|^2 \, dt, \quad z^1 \in \mathbb{R}^n,$$

where $z$ is the solution to the adjoint system (1.10). Since (1.1) is controllable, $N$ defines a norm on $\mathbb{R}^n$. Since all the norms are equivalent in finite dimension, there exists $C > 0$ such that

$$\left\|z^1\right\|^2 \leq C^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n.$$

It follows that

$$\begin{aligned} J(z^1) &\geq \frac{1}{2C^2}\left\|z^1\right\|^2 - \left|y^0 \cdot z(0)\right| \\ &\geq \frac{1}{2C^2}\left\|z^1\right\|^2 - \alpha\left\|z^1\right\|, \end{aligned}$$

where $\alpha = \left\|y^0\right\| e^{T\|A^*\|}$ does not depend on $z^1$. Therefore,

$$J(z^1) \xrightarrow[\|z^1\|\to+\infty]{} +\infty.$$

By Lemma 1.4.12, $J$ has a minimizer $z^1_{\text{opt}}$. Next, note that $J$ is differentiable on $\mathbb{R}^n$ with

$$DJ(\hat{z}^1)z^1 = \int_0^T B^*\hat{z}(t) \cdot B^* z(t)\, dt + y^0 \cdot z(0), \quad \forall \hat{z}^1, z^1 \in \mathbb{R}^n,$$

where $z$ (*resp.* $\hat{z}$) is the solution to the adjoint system (1.10) associated with $z^1$ (*resp.* $\hat{z}^1$). Since $z^1_{\text{opt}}$ is a minimizer of $J$, we have $DJ(z^1_{\text{opt}}) = 0$, that is

$$0 = \int_0^T B^* z_{\text{opt}}(t) \cdot B^* z(t)\, dt + y^0 \cdot z(0), \quad \forall z^1 \in \mathbb{R}^n.$$

This means that $u_{\text{opt}} = B^* z_{\text{opt}}$ is a null-control (see (1.29)). Let us finally prove that $u_{\text{opt}}$ is the unique null-control of minimal $L^2$-norm. Let $u \in L^2(0,T)^m$ be another null-control. Since $u$ and $u_{\text{opt}}$ are two null-controls, they both satisfy (1.29). Taking $z^1 = z^1_{\text{opt}}$ in (1.29), we obtain

$$\int_0^T (u(t) - u_{\text{opt}}(t)) \cdot u_{\text{opt}}(t)\, dt = 0.$$

It follows that

$$\|u\|^2_{L^2(0,T)^m} = \|u_{\text{opt}}\|^2_{L^2(0,T)^m} + \|u - u_{\text{opt}}\|^2_{L^2(0,T)^m}.$$

From this identity we see that $u_{\text{opt}}$ minimizes the $L^2$-norm among all possible null-controls and that it is the only one. □

## 1.5 Controls with constraints

In this section we will look for controls $u \in L^2(0,T)^m$ that satisfy in addition the constraint

$$u(t) \in U \quad \text{a.e. } t \in (0,T), \tag{1.30}$$

where $U$ is a fixed nonempty subset of $\mathbb{R}^m$. Let us first point out that we have already encountered controls that satisfy some constraints, see Remarks 1.1.6 and 1.3.5. In this section we provide some elements of the general theory for systems with constrained controls.

### 1.5.1 Sufficient conditions for large times

**Definition 1.5.1.** Let $C \subset \mathbb{R}^n$ be the set of elements $y^0 \in \mathbb{R}^n$ such that there exist $T > 0$ and $u \in L^2(0,T)^m$ with (1.30) such that the corresponding solution $y$ to (1.1) satisfies $y(T) = 0$. We say that the constrained system (1.1)-(1.30) is:

(i) null-controllable if $C = \mathbb{R}^n$.

(ii) locally null-controllable if $0 \in \mathring{C}$, where $\mathring{C}$ denotes the interior of the set $C$.

*Remark* 1.5.2. Observe that the time of control depends on the initial data in these definitions.

Let us start by investigating what the controllability of the unconstrained system (1.1) implies for the controllability of the constrained system (1.1)-(1.30).

**Theorem 1.5.3.** *Assume that* $0 \in \mathring{U}$. *The following statements are equivalent:*

*(i) The system* (1.1) *is controllable.*

*(ii) The system* (1.1)-(1.30) *is locally null-controllable.*

*Proof.* (i) $\Longrightarrow$ (ii). Assume that (1.1) is controllable. Then, by Theorem 1.4.1, there exists a control $u_{\text{opt}}$ with

$$\|u_{\text{opt}}(t)\| \leq M \left\|y^0\right\|, \quad \forall t \in [0,T],$$

for some $M > 0$ that depends only on $A, B$ and $T$. Since $0 \in \mathring{U}$ by assumption, there exists $r > 0$ such that, for every $u \in \mathbb{R}^m$, if $\|u\| < r$ then $u \in U$. Therefore, if $y^0$ is small enough, say $\left\|y^0\right\| < r/M$, then $u_{\text{opt}}(t) \in U$ for every $t \in [0,T]$ and $0 \in \mathring{C}$.

(ii) $\Longrightarrow$ (i). Conversely, assume that (1.1) is not controllable, that is

$$\operatorname{rank}(B|AB|\cdots|A^{n-1}B) < n.$$

Thus, there exists a non zero vector $\xi \in \mathbb{R}^n$ such that

$$\xi^* A^k B = 0, \quad \forall k \in \{0, 1, \ldots, n-1\}.$$

Using the Cayley-Hamilton theorem it follows that

$$\xi^* A^k B = 0, \quad \forall k \in \{0, 1, \ldots\}.$$

Thus,

$$\xi^* e^{tA} B = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \xi^* A^k B = 0, \quad \forall t \in \mathbb{R}.$$

Let now $y^0 \in C$. By definition, there exist $T > 0$ and $u \in L^2(0,T)^m$ such that

$$0 = e^{TA} y^0 + \int_0^T e^{(T-t)A} Bu(t)\, dt,$$

or, equivalently,

$$0 = y^0 + \int_0^T e^{-tA} Bu(t)\, dt.$$

Taking the inner product of this identity with $\xi$ we obtain that

$$\xi \cdot y^0 = 0.$$

Since this is true for any $y^0 \in C$, this shows that $C \subset \xi^\perp$. But $\xi^\perp$ is a vectorial space, which is not $\mathbb{R}^n$ since $\xi \neq 0$, and therefore its interior is empty. It follows that $C$ has an empty interior as well, so that $0 \notin \mathring{C}$. □

Let us now give an easy but interesting sufficient condition for the null-controllability of (1.1)-(1.30).

**Theorem 1.5.4.** *Assume that* $0 \in \mathring{U}$ *and:*

*(i) System* (1.1) *is controllable.*

*(ii) A is stable (that is,* $e^{tA} y^0 \to 0$ *as* $t \to +\infty$ *for every* $y^0 \in \mathbb{R}^n$*).*

*Then, the system* (1.1)-(1.30) *is null-controllable.*

*Proof.* By Theorem 1.5.3 we have $0 \in \mathring{C}$. Thus, there exists $r > 0$ such that, for every $y^0 \in \mathbb{R}^n$, if $\|y^0\| < r$, then $y^0 \in C$. Let $y^0 \in \mathbb{R}^n$ be fixed. Since $A$ is stable, we have

$$e^{tA}y^0 \xrightarrow[t\to+\infty]{} 0.$$

Therefore, there exists $T_1 > 0$ (large enough and depending on $y^0$) such that

$$\left\|e^{T_1A}y^0\right\| < r.$$

It follows that $e^{T_1A}y^0 \in C$. By definition of $C$, there exist $T_2 > 0$ and $u_2 \in L^2(T_1, T_1+T_2)^m$, with $u_2(t) \in U$ for a.e. $t \in (T_1, T_1 + T_2)$, such that the solution $y_2$ to

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}y_2 & = & Ay_2 + Bu_2, \quad t \in (T_1, T_1 + T_2), \\ y_2(T_1) & = & e^{T_1A}y^0, \end{array}\right.$$

satisfies $y_2(T_1 + T_2) = 0$. Thus, we see that the control defined by

$$u(t) = \left\{\begin{array}{cl} 0 & \text{for } t \in (0, T_1), \\ u_2(t) & \text{for } t \in (T_1, T_1 + T_2), \end{array}\right.$$

satisfies (1.30) and brings the corresponding solution to (1.1) from $y^0$ to 0 in time $T_1+T_2$. □

In the case of bounded control sets, there is a complete characterization of the null-controllability (the proof is more complex though, see e.g. [Son98, Theorem 6] (applied to $-A$ and $-B$ instead of $A$ and $B$)):

**Theorem 1.5.5.** *Assume that* $0 \in \mathring{U}$ *and that* $U$ *is bounded. Then, the system* (1.1)-(1.30) *is null-controllable if, and only if, the following two conditions hold:*

*(i) The system* (1.1) *is controllable.*

*(ii)* $\operatorname{Re}\lambda \leq 0$ *for every eigenvalue* $\lambda \in \mathbb{C}$ *of* $A$.

We recall that $A$ is stable if, and only if, $\operatorname{Re}\lambda < 0$ for every eigenvalue $\lambda \in \mathbb{C}$ of $A$ (see e.g. [Zab08, Theorem I.2.3]). Therefore the condition (ii) of Theorem 1.5.4 is stronger than the condition (ii) of Theorem 1.5.5.

### 1.5.2 Time-optimal problems

In the previous section we provided some sufficient conditions to ensure the null-controllability of (1.1)-(1.30) for large enough times. Therefore, it is natural to address the problem of finding the best time possible and a possible corresponding control.

#### 1.5.2.1 Existence of time-optimal controls

**Definition 1.5.6** (Reachable set)**.** For $y^0 \in \mathbb{R}^n$ and $T > 0$, let $R_T(y^0) \subset \mathbb{R}^n$ be the set of elements $y^1 \in \mathbb{R}^n$ such that there exists $u \in L^2(0,T)^m$ with (1.30) such that the corresponding solution $y$ to (1.1) satisfies $y(T) = y^1$. According to (1.2) it is the set of all elements

$$e^{TA}y^0 + \int_0^T e^{(T-t)A}Bu(t)\,dt, \tag{1.31}$$

for $u \in L^2(0,T)^m$ with (1.30). For $T = 0$ we naturally set $R_0(y^0) = \left\{y^0\right\}$.

**Proposition 1.5.7** (Properties of the reachable set)**.** *Assume that $U$ is compact. Let $y^0 \in \mathbb{R}^n$ be fixed. Then,*

*(i) $R_T(y^0)$ is compact and convex for every $T \geq 0$.*

*(ii) $R_T(y^0)$ varies continuously with respect to $T$. More precisely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $T_1, T_2 \geq 0$, if $|T_1 - T_2| < \delta$ then*

$$d(R_{T_1}(y^0), R_{T_2}(y^0)) \leq \varepsilon,$$

*where $d(X_1, X_2)$ denotes the Hausdorff distance between the closed subsets $X_1 \subset \mathbb{R}^n$ and $X_2 \subset \mathbb{R}^n$, that is $d(X_1, X_2) = \max\left\{\sup_{x_1 \in X_1} d(x_1, X_2), \sup_{x_2 \in X_2} d(X_1, x_2)\right\}$.*

For a proof of this proposition we refer to [LM67, Theorem 2.1] if $U$ is convex and [LM67, Theorem 2.1A] for the general case.

**Theorem 1.5.8** (Existence of time-optimal controls)**.** *Assume that $U$ is compact. Let $y^0, y^1 \in \mathbb{R}^n$ be fixed. Assume that there exists $T \geq 0$ such that $y^1 \in R_T(y^0)$. Then, the set $\left\{T \geq 0,\ y^1 \in R_T(y^0)\right\}$ has a minimum $T_{\min} \geq 0$. By definition, this means that $T_{\min} = 0$ if, and only if, $y^1 = y^0$ and, if $T_{\min} > 0$, this means that there exists $u \in L^2(0, T_{\min})^m$ with* (1.30) *such that the corresponding solution $y$ to* (1.1) *satisfies $y(T_{\min}) = y^1$. Such a $u$ is called a time-optimal control.*

*Proof.* Let

$$E = \left\{T \geq 0,\ y^1 \in R_T(y^0)\right\}.$$

By assumption, $E$ is not empty. To prove that $E$ has a minimum we show that it is closed. Let then $T_k \in E$, $k \in \mathbb{N}$, and $T \in \mathbb{R}$ be such that $T_k \to T$ as $k \to +\infty$. We have to prove that $T \in E$. Clearly, $T \geq 0$. Let us now prove that $y^1 \in R_T(y^0)$. Since $R_T(y^0)$ is closed (see Proposition 1.5.7), it is equivalent to prove that $d(y^1, R_T(y^0)) = 0$. Let $\varepsilon > 0$. We have

$$d(y^1, R_T(y^0)) \leq d(y^1, R_{T_k}(y^0)) + d(R_{T_k}(y^0), R_T(y^0)).$$

Since $y^1 \in R_{T_k}(y^0)$ by definition of $T_k$, we have $d(y^1, R_{T_k}(y^0)) = 0$. Now, since $T_k \to T$, by continuity (see Proposition 1.5.7) there exists $k \in \mathbb{N}$ large enough so that $d(R_{T_k}(y^0), R_T(y^0)) \leq \varepsilon$. Therefore, we have proved that $d(y^1, R_T(y^0)) \leq \varepsilon$ for every $\varepsilon > 0$, that is $d(y^1, R_T(y^0)) = 0$. □

#### 1.5.2.2 Maximum principle

Before proving the so-called Pontryagin maximum principle, we establish some properties of time-optimal controls.

**Definition 1.5.9** (Extremal control). Let $y^0 \in \mathbb{R}^n$ and $T > 0$ be fixed. A function $u \in L^2(0,T)^m$ is called an extremal control if $u$ satisfies (1.30) and the corresponding solution $y$ to (1.1) satisfies $y(T) \in \partial R_T(y^0)$.

**Theorem 1.5.10** (Time-optimal controls are extremal). *Assume that $U$ is compact. Let $y^0, y^1 \in \mathbb{R}^n$ be such that $y^1 \neq y^0$. Assume that there exists $T > 0$ such that $y^1 \in R_T(y^0)$. Let $T_{\min} > 0$ be the optimal time and let $u \in L^2(0, T_{\min})^m$ be a time-optimal control (whose existences are guaranteed by Theorem 1.5.8). Then, $u$ is an extremal control.*

We will need the following result from convex analysis (for a proof, see e.g. [Zab08, Theorem III.3.5])

**Lemma 1.5.11** (Hyperplane separation theorem). *Let $C \subset \mathbb{R}^n$ be a convex subset and $a \in \mathbb{R}^n$. There exists $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ such that*

$$\xi \cdot y \leq \xi \cdot a, \quad \forall y \in C$$

*if, and only if, $a \notin \mathring{C}$.*

*Proof of Theorem 1.5.10.* We have to show that $y^1 \in \partial R_{T_{\min}}(y^0)$. Since $R_{T_{\min}}(y^0)$ is a closed convex (see Proposition 1.5.7), by Lemma 1.5.11, it is equivalent to prove that there exists $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ such that, for every $\hat{y}^1 \in R_{T_{\min}}(y^0)$,

$$\xi \cdot \hat{y}^1 \leq \xi \cdot y^1. \tag{1.32}$$

Let $T_k > 0$, $k \in \mathbb{N}^*$, be such that $T_k \to T_{\min}$ as $k \to +\infty$ with $T_k < T_{\min}$ for every $k \in \mathbb{N}^*$. Since $T_k < T_{\min}$, by definition of $T_{\min}$ we have

$$y^1 \notin R_{T_k}(y^0), \quad \forall k \in \mathbb{N}^*.$$

In particular $y^1 \notin \mathring{R_{T_k}}(y^0)$. Since $R_{T_k}(y^0)$ is convex, by Lemma 1.5.11 there exists $\xi_k \in \mathbb{R}^n$ with $\xi_k \neq 0$ such that, for every $w^1 \in R_{T_k}(y^0)$,

$$\xi_k \cdot w^1 \leq \xi_k \cdot y^1. \tag{1.33}$$

Since $\xi_k \neq 0$, we can assume that $\|\xi_k\| = 1$. Since $(\xi_k)_k$ is now a bounded sequence, we can extract a subsequence (still denoted by $(\xi_k)_k$) such that $\xi_k \to \xi$ as $k \to +\infty$ for some $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ (as $\|\xi\| = 1$). Let $\hat{y}^1 \in R_{T_{\min}}(y^0)$ be fixed. Take a sequence $(\hat{y}^1_j)_j$ such that $\hat{y}^1_j \to \hat{y}^1$ as $j \to +\infty$ with $\hat{y}^1_j \in R_{T_{k_j}}(y^0)$ for every $j \in \mathbb{N}^*$ for some $k_j \to +\infty$ as $j \to +\infty$. Such a sequence exists because of the continuity of the reachable sets (see

Proposition 1.5.7). Indeed, since $T_j \to T_{\min}$ as $j \to +\infty$, for $j$ large enough there exists $k_j$, with $k_j \to +\infty$ as $j \to +\infty$, such that

$$d(R_{T_{k_j}}(y^0), \hat{y}^1) < \frac{1}{j}.$$

Therefore, there exists $\hat{y}^1_j \in R_{T_{k_j}}(y^0)$ such that

$$d(\hat{y}^1_j, \hat{y}^1) < \frac{1}{j}.$$

Finally, taking $w^1 = \hat{y}^1_j$ in (1.33) and passing to the limit as $j \to +\infty$, we obtain (1.32). □

Thanks to Theorem 1.5.10 we can now focus on the notion of extremal control.

**Theorem 1.5.12** (Pontryagin maximum principle). *Assume that $U$ is compact. Let $y^0 \in \mathbb{R}^n$, $T > 0$ and $u \in L^2(0,T)^m$. The following statements are equivalent:*

*(i) $u$ is an extremal control.*

*(ii) There exists $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ such that the corresponding solution $z$ to the adjoint system*

$$\begin{cases} -\dfrac{d}{dt}z &= A^*z, \quad t \in (0,T), \\ z(T) &= z^1, \end{cases}$$

*satisfies*

$$B^*z(t) \cdot u(t) = \max_{u \in U} B^*z(t) \cdot u \quad \text{a.e. } t \in (0,T). \tag{1.34}$$

*Proof.* By definition, $u$ is an extremal control if, and only if, $y(T) \in \partial R_T(y^0)$, where $y$ is the corresponding solution to (1.1). Since $R_T(y^0)$ is a closed convex (see Proposition 1.5.7), by Lemma 1.5.11 this is equivalent to the existence of $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ such that, for every $\hat{y}^1 \in R_T(y^0)$,

$$z^1 \cdot \hat{y}^1 \leq z^1 \cdot y(T).$$

Recalling (1.31), this inequality is equivalent to

$$z^1 \cdot \int_0^T e^{(T-t)A} B\left(\hat{u}(t) - u(t)\right) dt \leq 0,$$

that is,

$$\int_0^T B^*z(t) \cdot \left(\hat{u}(t) - u(t)\right) dt \leq 0, \tag{1.35}$$

for every $\hat{u} \in L^2(0,T)^m$ with $\hat{u}(t) \in U$ for a.e. $t \in (0,T)$. Therefore, if $u$ satisfies (1.34) then, in particular,

$$B^*z(t) \cdot u(t) \geq B^*z(t) \cdot \hat{u}(t) \quad \text{a.e. } t \in (0,T),$$

for every $\hat{u} \in L^2(0,T)^m$ with $\hat{u}(t) \in U$ for a.e. $t \in (0,T)$. Integrating this inequality, we obtain (1.35). Conversely, assume that (1.35) holds and let us prove that $u$ satisfies (1.34). It is clear that there exists a function $w : (0,T) \to U$ such that

$$\max_{u \in U} B^* z(t) \cdot u = B^* z(t) \cdot w(t), \quad \text{a.e. } t \in (0,T).$$

It can be proved that $w$ can even be chosen so that $w \in L^2(0,T)^m$ (see e.g. [LM67, Lemma 1.2A and 1.3A]). In particular,

$$B^* z(t) \cdot w(t) \geq B^* z(t) \cdot u(t), \quad \text{a.e. } t \in (0,T),$$

and we can integrate this inequality to obtain the reverse inequality of (1.35) for $\hat{u} = w$. As a result, $t \longmapsto B^* z(t) \cdot w(t) - B^* z(t) \cdot u(t)$ is a positive function whose integral is zero and therefore is itself equal to zero. □

#### 1.5.2.3 Bang-bang controls

**Theorem 1.5.13** (Bang-bang principle)**.** *Assume that $U$ is compact. Let $y^0 \in \mathbb{R}^n$, $T > 0$ and $u \in L^2(0,T)^m$ Assume that* (1.1) *is controllable. If $u$ is an extremal control, then*

$$u(t) \in \partial U, \quad \text{a.e. } t \in (0,T).$$

**Lemma 1.5.14.** *Let $U$ be a closed subset of $\mathbb{R}^n$. Let $q \in \mathbb{R}^m$ and define the function $f : U \longrightarrow \mathbb{R}$ by $f(u) = q \cdot u$. Assume that $q \neq 0$. If $u_0 \in U$ is a point of local maximum of $f$, then $u_0 \in \partial U$.*

*Proof.* Let $u_0 \in U$ be a point of local maximum of $f$. Assume that $u_0 \in \mathring{U}$. Then, there exists $\varepsilon > 0$ such that $u_0 + \varepsilon q \in U$. But

$$f(u_0 + \varepsilon q) = f(u_0) + \varepsilon \left\| q \right\|^2 > f(u_0),$$

where the inequality is strict because $q \neq 0$. This is a contradiction with the local maximality of $u_0$. □

**Lemma 1.5.15** (Number of switches)**.** *Assume that* (1.1) *is controllable. Then, for every $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ and $T > 0$, the set*

$$Z = \{t \in (0,T),\ B^* z(t) = 0\}$$

*is finite.*

*Proof.* Assume that $Z$ is infinite. Then, by analyticity of $z$ we obtain

$$B^* z(t) = 0, \quad \forall t \in [0,T].$$

The controllability of (1.1) then implies that $z^1 = 0$ (see Theorem 1.2.3), a contradiction. □

*Proof of Theorem 1.5.13.* By Theorem 1.5.12, there exists $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ such that $z(t) = e^{(T-t)A^*} z^1$ satisfies

$$B^* z(t) \cdot u(t) = \max_{u \in U} B^* z(t) \cdot u \quad \text{a.e. } t \in (0,T).$$

For every $t \in (0,T)$ and $u \in U$ we define $f_t(u) = B^* z(t) \cdot u$. Observe that $B^* z = 0$ only on a set of zero measure by Lemma 1.5.15. Therefore, the conclusion follows from Lemma 1.5.14. □

*Remark* 1.5.16. In the case $m = 1$ and $U = [a,b]$ $(a < b)$, we see that the function $f$ of Lemma 1.5.14 only has one maximum, which is attained at $u = b$ if $q > 0$ and at $u = a$ if $q < 0$. Therefore, in this case, if $u \in L^2(0,T)$ is an extremal control, then, for a.e. $t \in (0,T)$,

$$u(t) = \begin{cases} b & \text{if } B^* z(t) > 0, \\ a & \text{if } B^* z(t) < 0, \end{cases}$$

for some $z^1 \neq 0$. This explains the terminology "bang-bang".

## 1.6 Bibliographical notes

**The Fattorini-Hautus test.** Theorem 1.3.11 is misleadingly known as the Hautus test (or sometimes Popov–Belevitch–Hautus test) despite it was actually proved three years earlier in 1966 by H. Fattorini and, moreover, in a general abstract setting (that trivially includes finite dimensional systems).

**Control with constraints.** For additional material on constrained controllability and time-optimal problems we refer to [LM67, Chapter 2], [Son98, Section 3.6 and Chapter 4] and the references therein. Let us also mention that other type of constraints such as positivity of the controls are also discussed in [Zab08, Part I, Chapter 4].

**Time-dependent O.D.E.s.** We refer to [Cor07, Chapter 1] for controllability conditions for time-dependent linear O.D.E.s. Notably, one can find the result of L. Silverman and H. Meadows and related propositions that extend the Kalman rank condition.

**Nonlinear O.D.E.s.** Good material for an introduction to the controllability of nonlinear systems can be found in [Cor07, Chapter 3], [Son98, Chapter 4] and [Zab08, Part II, Chapter 1].

# Chapter 2

# Controllability of the heat equation

## 2.1 Background on the heat equation

In this chapter we consider the heat equation on a nonempty bounded open subset $\Omega \subset \mathbb{R}^N$ of class $C^2$:

$$\begin{cases} \partial_t y - \Delta y = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where:

- $T > 0$ is the time of control,
- $y$ is the state $y^0$ is the initial data,
- $u$ is the control,
- $\omega \subset \Omega$ localizes in space the control (we assume that $\omega$ is a nonempty open subset),
- $\mathbb{1}_\omega$ is the characteristic function of the set $\omega$, that is the function defined by

$$\mathbb{1}_\omega(x) = \begin{cases} 1 & \text{if } x \in \omega, \\ 0 & \text{if } x \notin \omega. \end{cases}$$

We will also use the same notation to denote the operator $\mathbb{1}_\omega : L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$\mathbb{1}_\omega u(x) = \begin{cases} u(x) & \text{if } x \in \omega, \\ 0 & \text{if } x \notin \omega. \end{cases}$$

Note that $\mathbb{1}_\omega$ is a bounded linear operator with $\|\mathbb{1}_\omega\|_{\mathcal{L}(L^2(\Omega))} \leq 1$.

For a function $y$ of $(t,x)$ we will use the notation $y(t)$ to denote the function $y(t) : x \longmapsto y(t,x)$.

Our presentation will be based on the following fundamental result:

**Theorem 2.1.1** (Spectral decomposition). *There exists an orthonormal basis of $L^2(\Omega)$ formed of eigenfunctions of the Dirichlet Laplacian $\Delta$. More precisely, there exist $\{\phi_k\}_{k\in\mathbb{N}^*} \subset H^2(\Omega)\cap H^1_0(\Omega)$ and $\{-\lambda_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}$ such that, for every $k \in \mathbb{N}^*$,*

$$\begin{cases} \Delta\phi_k &= -\lambda_k\phi_k \quad \text{in } \Omega, \\ \phi_k &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

*with*

$$\langle \phi_k, \phi_\ell\rangle_{L^2(\Omega)} = \delta_{k\ell}, \quad \forall k,\ell \in \mathbb{N}^*, \qquad w = \sum_{k=1}^{+\infty} \langle w, \phi_k\rangle_{L^2}\phi_k, \quad \forall w \in L^2(\Omega),$$

*and*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots, \qquad \lambda_k \xrightarrow[k\to+\infty]{} +\infty.$$

Let us do a little digression to motivate the notion of solution we will introduce for (2.1). Assume that $y \in C^2([0,T]\times\overline{\Omega})$ is a classical solution to (2.1), that is, the equation, the boundary condition and the initial condition hold pointwisely (this implicitly requires that $u \in C^0_c([0,T]\times\omega)$). Then, multiplying (2.1) by $\phi_k$ and integrating over $\Omega$ we obtain the O.D.E.

$$\begin{cases} y_k'(t) + \lambda_k y_k(t) = f_k(t), \quad t \in (0,T), \\ y_k(0) = y_k^0, \end{cases}$$

where

$$y_k(t) = \langle y(t), \phi_k\rangle_{L^2}, \quad y_k^0 = \left\langle y^0, \phi_k\right\rangle_{L^2}, \quad f_k(t) = \langle \mathbb{1}_\omega u(t), \phi_k\rangle_{L^2}.$$

Thus,

$$y_k(t) = e^{-\lambda_k t}y_k^0 + \int_0^t e^{-\lambda_k(t-s)}f_k(s)\,ds, \quad \forall t \in [0,T].$$

Since $y(t) \in L^2(\Omega)$, we have

$$y(t) = \sum_{k=1}^{+\infty} \langle y(t), \phi_k\rangle_{L^2}\phi_k.$$

Therefore,

$$\begin{aligned} y(t) &= \sum_{k=1}^{+\infty} e^{-\lambda_k t}y_k^0\phi_k + \sum_{k=1}^{+\infty}\left(\int_0^t e^{-\lambda_k(t-s)}f_k(s)\,ds\right)\phi_k \\ &= \sum_{k=1}^{+\infty} e^{-\lambda_k t}y_k^0\phi_k + \int_0^t \left(\sum_{k=1}^{+\infty} e^{-\lambda_k(t-s)}f_k(s)\phi_k\right) ds. \end{aligned}$$

**Definition 2.1.2** (Mild solution)**.** Let $T > 0$, $y^0 \in L^2(\Omega)$ and $u \in L^2(0,T;L^2(\Omega))$. The function $y : (0,T) \times \Omega \longrightarrow \mathbb{R}$ defined for every $t \in [0,T]$ by

$$y(t) = S(t)y^0 + \int_0^t S(t-s)\mathbb{1}_\omega u(s)\,ds, \tag{2.2}$$

where

$$S(t)y^0 = \sum_{k=1}^{+\infty} e^{-\lambda_k t} \langle y^0, \phi_k \rangle_{L^2} \phi_k, \tag{2.3}$$

is called the (mild) solution to (2.1).

Note that, for every $t \in [0,T]$, the series in (2.3) converges in $L^2(\Omega)$ with

$$\left\| S(t)y^0 \right\|_{L^2(\Omega)} \leq \left\| y^0 \right\|_{L^2(\Omega)}, \quad \forall y^0 \in L^2(\Omega).$$

On the other hand, using this estimate and the Cauchy-Schwarz inequality, we see that, for every $t \in [0,T]$, the map $s \mapsto S(t-s)\mathbb{1}_\omega u(s)$ in (2.2) belongs to $L^1(0,t;L^2(\Omega))$ and

$$\left\| \int_0^t S(t-s)\mathbb{1}_\omega u(s)\,ds \right\|_{L^2(\Omega)} \leq \sqrt{T}\,\|u\|_{L^2(0,T;L^2(\Omega))}, \quad \forall u \in L^2(0,T;L^2(\Omega)).$$

Therefore, $y$ is well-defined and we have the estimate

$$\|y(t)\|_{L^2(\Omega)} \leq \left\| y^0 \right\|_{L^2(\Omega)} + \sqrt{T}\,\|u\|_{L^2(0,T;L^2(\Omega))}, \quad \forall t \in [0,T]. \tag{2.4}$$

*Remark* 2.1.3. Formally, (2.1) can be recast as an infinite dimensional O.D.E.in the Hilbert space $H = L^2(\Omega)$:

$$\begin{cases} \dfrac{d}{dt}y &= Ay + Bu, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases}$$

where $A : D(A) \subset H \longrightarrow H$ is the Dirichlet Laplacian operator:

$$A = \Delta, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

and $B \in \mathcal{L}(H,U)$ with control space $U = L^2(\Omega)$ is simply $B = \mathbb{1}_\omega$. Then, (2.2) is nothing but the Duhamel's formula where $S(t)$ is the generalization of the exponential of a matrix (it is the so-called semigroup generated by $\Delta$).

All along this chapter we choose a presentation that is based on the explicit Fourier representation (2.2) for the solution to the heat equation (and of its adjoint system, see (2.8) below) since it does not require any particular knowledge in PDEs, semigroup nor spectral theory.

## 2.2 Controllability and duality

**Definition 2.2.1** (Controllability)**.** We say that (2.1) is:

(i) exactly controllable in time $T$ if, for every $y^0, y^1 \in L^2(\Omega)$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution $y$ to (2.1) satisfies

$$y(T) = y^1.$$

(ii) null-controllable in time $T$ if the above property holds for $y^1 = 0$.

(iii) approximately controllable in time $T$ if, for every $\varepsilon > 0$ and every $y^0, y^1 \in L^2(\Omega)$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution $y$ to (2.1) satisfies

$$\left\| y(T) - y^1 \right\|_{L^2(\Omega)} \leq \varepsilon.$$

*Remark* 2.2.2. The notion of exact controllability is not relevant for the heat equation because of its regularizing effect. Indeed, since $\mathbb{1}_\omega u = 0$ on the open set $\Omega \backslash \overline{\omega}$ for any $u$, the local parabolic regularity implies that the solution $y$ to (2.1) satisfies $y(\varepsilon) \in C^\infty(\Omega\backslash\overline{\omega})$ as soon as $\varepsilon > 0$. It follows that a target $y^1 \notin C^\infty(\Omega\backslash\overline{\omega})$ will never be reached, whatever the control time $T$ is.

*Remark* 2.2.3. The investigation of the controllability properties for (2.1) is difficult because $\omega$ is just a subset of $\Omega$. In the case $\omega = \Omega$, it is easy to see that (2.1) is null-controllable in time $T$ for every $T > 0$ (and thus approximately controllable too, see Remark 2.2.5 below). Indeed, if $y^0$ is smooth then we take any smooth function $y$ with $y(0) = y^0$ and $y(T) = 0$ and we simply set $u = \partial_t y - \Delta y$, just as in Example 1.1.2. If $y^0$ is not smooth, we just wait a little bit (with $u = 0$ during that time) to obtain a new initial data that is now smooth enough thanks to the regularizing effect and we apply the previous argument.

We now proceed as in the finite dimensional case. We introduce the linear operators

$$\begin{array}{rcccl} F_T & : & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & & y^0 & \longmapsto & \overline{y}(T), \end{array}$$

where $\overline{y}$ is the solution to the equation (2.1) with $u = 0$, and

$$\begin{array}{rcccl} G_T & : & L^2(0,T;L^2(\Omega)) & \longrightarrow & L^2(\Omega) \\ & & u & \longmapsto & \widehat{y}(T), \end{array}$$

where $\hat{y}$ is the solution to the equation (2.1) with $y^0 = 0$. With these notations, we can restate the different notions of controllability as follows:

(i) (2.1) is null-controllable in time $T$ if, and only if,

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T.$$

(ii) (2.1) is approximately controllable in time $T$ if, and only if,

$$\overline{\operatorname{Im} G_T} = L^2(\Omega).$$

Since $F_T$ and $G_T$ are bounded linear operators thanks to (2.4), by duality we obtain:

(i) (2.1) is null-controllable in time $T$ if, and only if, (see e.g. [TW09, Proposition 12.1.2])

$$\exists C > 0, \quad \left\|F_T^* z^1\right\|_{L^2(\Omega)}^2 \leq C^2 \left\|G_T^* z^1\right\|_{L^2(\Omega)}^2, \quad \forall z^1 \in L^2(\Omega).$$

(ii) (2.1) is approximately controllable in time $T$ if, and only if,

$$\ker G_T^* = \{0\}.$$

We continue to mimic the procedure developed in the finite dimensional case by computing the adjoint operators of $F_T$ and $G_T$. To this end we start by multiplying formally the equation (2.1) by a smooth function $z$ and we integrate over $(0,T) \times \Omega$. This leads to the following fundamental relation

$$\left\langle y(T), z^1 \right\rangle_{L^2(\Omega)} - \left\langle y^0, z(0) \right\rangle_{L^2(\Omega)} = \int_0^T \langle u(t), \mathbb{1}_\omega z(t) \rangle_{L^2(\Omega)} \, dt, \tag{2.5}$$

if $z$ is the solution to the so-called adjoint system:

$$\begin{cases} -\partial_t z - \Delta z = 0 & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega, \\ z(T) = z^1 & \text{in } \Omega. \end{cases} \tag{2.6}$$

Note that the equation in (2.6) is backward in time and therefore a priori ill-posed. However, observe that we consider a final condition in (2.6) and not an initial condition. Thus, the simple change of variable $t \mapsto T - t$ shows that $z(t) = \tilde{z}(T-t)$, where $\tilde{z}$ is the solution to

$$\begin{cases} \partial_t \tilde{z} - \Delta \tilde{z} = 0 & \text{in } (0,T) \times \Omega, \\ \tilde{z} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \tilde{z}(0) = z^1 & \text{in } \Omega. \end{cases} \tag{2.7}$$

As in Definition 2.1.2, by solution to (2.7), we mean

$$\tilde{z}(t) = \sum_{k=1}^{+\infty} e^{-\lambda_k t} \left\langle z^1, \phi_k \right\rangle_{L^2} \phi_k, \quad \forall t \in [0,T]. \tag{2.8}$$

The relation (2.5) can then be checked explicitly using (2.2) and (2.8). This relation makes the computations of $F_T^*$ and $G_T^*$ immediate. Therefore, we have obtained the following result:

**Theorem 2.2.4** (Duality). *Let $T>0$.*

*(i)* (2.1) *is null-controllable in time $T$ if, and only if, there exists $C>0$ such that*

$$\|z(0)\|^2_{L^2(\Omega)} \leq C^2 \int_0^T \|\mathbb{1}_\omega z(t)\|^2_{L^2(\Omega)}, \quad \forall z^1 \in L^2(\Omega), \tag{2.9}$$

*where $z$ is the solution to the adjoint system* (2.6).

*(ii)* (2.1) *is approximately controllable in time $T$ if, and only if,*

$$\forall z^1 \in L^2(\Omega), \quad \left(\mathbb{1}_\omega z(t) = 0, \quad \forall t \in [0,T]\right) \Longrightarrow z^1 = 0, \tag{2.10}$$

*where $z$ is the solution to the adjoint system* (2.6).

*Remark* 2.2.5. For the equation (2.1), the null-controllability is a stronger property than the approximate controllability. Indeed, the adjoint system (2.6) satisfies the so-called backward uniqueness, namely,

$$z(0) = 0 \Longrightarrow z^1 = 0.$$

This shows that (2.9) implies (2.10). Therefore, by Theorem 2.2.4, if (2.1) is null-controllable in time $T$, then (2.1) is approximately controllable in time $T$.

## 2.3 Approximate controllability

The goal of this section is to prove the following result.

**Theorem 2.3.1** (Approximate controllability). (2.1) *is approximately controllable in time $T$ for every $T>0$.*

In this section we will need to consider the distinct eigenvalues of the Dirichlet Laplacian $\Delta$. They will be denoted by $\left\{-\widehat{\lambda}_k\right\}_{k\in\mathbb{N}^*}$ and assumed to be ordered as

$$0 < \widehat{\lambda}_1 < \widehat{\lambda}_2 < \ldots.$$

Note that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$ since, by definition, we have $\widehat{\lambda}_k \geq \lambda_k$. For every $k \in \mathbb{N}^*$, let then $P_k : L^2(\Omega) \longrightarrow L^2(\Omega)$ be the orthogonal projection on $\ker(-\widehat{\lambda}_k - \Delta)$. Thus, we have

$$P_k z^1 = \sum_{j:\, \lambda_j = \widehat{\lambda}_k} \left\langle z^1, \phi_j \right\rangle_{L^2} \phi_j, \quad z^1 \in L^2(\Omega).$$

Let us restate all the properties we need from $\{\phi_j\}_{j\in\mathbb{N}^*}$ in terms of $\{P_k\}_{k\in\mathbb{N}^*}$. Clearly, $P_k$ is a bounded linear operator. Observe that

$$\begin{cases} P_k^* = P_k, \\ P_k P_\ell = \delta_{k\ell} P_k, \quad \forall \ell \in \mathbb{N}^*. \end{cases}$$

In particular, we have the following useful fact that will be used ceaselessly to compute the square of the $L^2$-norm of various series:

$$\left\|\sum_{k=1}^{+\infty} \alpha_k P_k z^1\right\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} |\alpha_k|^2 \left\|P_k z^1\right\|^2_{L^2(\Omega)},$$

where $z^1 \in L^2(\Omega)$ and $(\alpha_k)_{k\in\mathbb{N}^*}$ are such that one of the series converges. Finally, note that, for every $K \in \mathbb{N}^*$, there exists $J_K \in \mathbb{N}^*$ with $J_K \geq K$ such that, for every $z^1 \in L^2(\Omega)$ and $t \in [0, +\infty)$ we have

$$\sum_{k=1}^{K} e^{-\widehat{\lambda}_k t} P_k z^1 = \sum_{j=1}^{J_K} e^{-\lambda_j t} \langle z^1, \phi_j \rangle_{L^2} \phi_j.$$

Passing to the limit $K \to +\infty$, this shows that

$$z^1 = \sum_{k=1}^{+\infty} P_k z^1, \quad z^1 \in L^2(\Omega), \tag{2.11}$$

and that the solution $\tilde{z}$ to (2.7) is

$$\tilde{z}(t) = \sum_{k=1}^{+\infty} e^{-\widehat{\lambda}_k t} P_k z^1, \quad \forall t \in [0, +\infty). \tag{2.12}$$

Note in particular that (2.11) implies that, for every $z^1 \in L^2(\Omega)$,

$$\left(P_k z^1 = 0, \quad \forall k \in \mathbb{N}^*\right) \quad \Longleftrightarrow \quad z^1 = 0. \tag{2.13}$$

Let us now give two important lemma for the proof of Theorem 2.3.1.

**Lemma 2.3.2.** *For every $z^1 \in L^2(\Omega)$, the solution $\tilde{z}$ to (2.7) is analytic on $(0, +\infty)$.*

*Remark* 2.3.3. It follows from Lemma 2.3.2 that (2.10) is equivalent to

$$\forall z^1 \in L^2(\Omega), \quad \Big(\mathbb{1}_\omega \tilde{z}(t) = 0, \quad \forall t \in [0, +\infty)\Big) \Longrightarrow z^1 = 0.$$

Thus, the approximate controllability of (2.1) does not depend on the time of control $T$.

*Proof of Lemma 2.3.2.* From the expression (2.12) we can check that $\tilde{z}$ is infinitely differentiable on $(0, +\infty)$ with, for every $t \in (0, +\infty)$ and $j \in \mathbb{N}$,

$$\partial_t^j \tilde{z}(t) = \sum_{k=1}^{+\infty} (-\widehat{\lambda}_k)^j e^{-\widehat{\lambda}_k t} P_k z^1.$$

To prove that $\tilde{z}$ is analytic on $(0,+\infty)$ we recall that it is then sufficient to establish that

$$\forall [a,b] \subset (0,+\infty), \exists C>0, \quad \max_{t\in[a,b]} \frac{\left\|\partial_t^j \tilde{z}(t)\right\|_{L^2(\Omega)}}{j!} \leq C^{j+1}, \quad \forall j \in \mathbb{N}. \tag{2.14}$$

Indeed, since $\tilde{z} \in C^\infty(0,+\infty)$ we can write the Taylor expansion of $\tilde{z}$ up to any order. More precisely, for every $t_0 \in (0,+\infty)$, let $\rho > 0$ be such that $[t_0-\rho, t_0+\rho] \subset (0,+\infty)$. Then, for every $n \in \mathbb{N}$ and for every $t \in \mathbb{R}$ with $|t-t_0|<\rho$, the remainder

$$R_n(t) = \tilde{z}(t) - \sum_{j=0}^{n} \frac{\partial_t^j \tilde{z}(t_0)}{j!}(t-t_0)^j,$$

satisfies

$$\|R_n(t)\|_{L^2(\Omega)} \leq \max_{\xi\in[t_0-\rho,t_0+\rho]} \left\|\partial_t^{n+1}\tilde{z}(\xi)\right\|_{L^2(\Omega)} \frac{|t-t_0|^{n+1}}{(n+1)!}.$$

Thus, using (2.14) with $[a,b]=[t_0-\rho,t_0+\rho]$ and taking $r>0$ such that $Cr<1$ we see that $R_n(t) \to 0$ as $n \to +\infty$ for every $|t-t_0|<r$. Let us now establish (2.14). Firstly, note that, for every $j \in \mathbb{N}^*$, we have

$$\max_{x>0} x^j e^{-x} = j^j e^{-j} \leq j!,$$

(the inequality can be proved by induction). Thus, for every $t>0$, we have

$$\begin{aligned} \left\|\partial_t^j \tilde{z}(t)\right\|_{L^2(\Omega)}^2 &= \left(\frac{1}{t}\right)^{2j} \sum_{k=1}^{+\infty} \left((\widehat{\lambda}_k t)^j e^{-\widehat{\lambda}_k t}\right)^2 \left\|P_k z^1\right\|_{L^2(\Omega)}^2 \\ &\leq \left(\frac{1}{t}\right)^{2j} (j!)^2 \left\|z^1\right\|_{L^2(\Omega)}^2 . \end{aligned}$$

This shows that (2.14) holds with $C=\max(1/a, \left\|z^1\right\|_{L^2(\Omega)})$. □

We will also need the following important result (for a proof see e.g. [Hör76, Theorem 7.5.1]):

**Lemma 2.3.4.** *Let $k \in \mathbb{N}^*$. Every $\phi \in \ker(-\widehat{\lambda}_k - \Delta)$ is analytic on $\Omega$.*

*Remark* 2.3.5. It follows from Lemma 2.3.4 that we have the following property:

$$\ker(-\widehat{\lambda}_k - \Delta) \cap \ker \mathbb{1}_\omega = \{0\}, \quad \forall k \in \mathbb{N}^*. \tag{2.15}$$

Observe that, formally, this is nothing but the Fattorini-Hautus test (1.20) with $A=\Delta$ and $B=\mathbb{1}_\omega$. Note as well that (2.15) is a necessary condition to the approximate controllability (just mimic the first part of the proof of Theorem 1.3.11). We will see below that this condition is also sufficient. It is the key property for the approximate controllability.

We are now going to provide two proofs of Theorem 2.3.1. The first proof right below is the classical proof that is presented in many textbooks.

*Proof of Theorem 2.3.1.* By Remark 2.3.3, we have to prove that

$$\forall z^1 \in L^2(\Omega), \quad \Big(\mathbb{1}_\omega \tilde{z}(t) = 0, \quad \forall t \in [0,+\infty)\Big) \Longrightarrow z^1 = 0,$$

where $\tilde{z}$ is the solution to (2.7). From the expression (2.12) of $\tilde{z}$ and since $\mathbb{1}_\omega$ is a bounded operator, we have $\mathbb{1}_\omega \tilde{z}(t) = 0$ for every $t \in [0,+\infty)$ if, and only if,

$$\sum_{k=1}^{+\infty} e^{-\widehat{\lambda}_k t} \mathbb{1}_\omega P_k z^1 = 0, \quad \forall t \in [0,+\infty). \tag{2.16}$$

Multiplying this identity by $e^{\widehat{\lambda}_1 t}$, this is equivalent to

$$\mathbb{1}_\omega P_1 z^1 + \sum_{k=2}^{+\infty} e^{-(\widehat{\lambda}_k - \widehat{\lambda}_1)t} \mathbb{1}_\omega P_k z^1 = 0, \quad \forall t \in [0,+\infty).$$

Since $\widehat{\lambda}_k - \widehat{\lambda}_1 \geq \widehat{\lambda}_2 - \widehat{\lambda}_1 > 0$ for every $k \geq 2$, taking the limit $t \to +\infty$, we obtain

$$\mathbb{1}_\omega P_1 z^1 = 0.$$

Since $P_1 z^1 \in \ker(-\widehat{\lambda}_1 - \Delta)$, (2.15) yields

$$P_1 z^1 = 0.$$

Coming back to (2.16) we obtain

$$\sum_{k=2}^{+\infty} e^{-\widehat{\lambda}_k t} \mathbb{1}_\omega P_k z^1 = 0, \quad \forall t \in [0,+\infty).$$

Multiplying this time by $e^{\widehat{\lambda}_2 t}$ and using the same arguments as before we obtain that $P_2 z^1 = 0$. By induction, we obtain

$$P_k z^1 = 0, \quad \forall k \in \mathbb{N}^*,$$

so that $z^1 = 0$ by (2.13). $\square$

Let us now present a different proof. It is slightly longer in this case but it has the advantage to be generalizable to more general parabolic equations than the heat equation. Note in particular that in the proof below it is not directly required to write the solution along the projections $P_k$ (which is a basis property and can be difficult to establish for general parabolic systems). In other words, the property (2.13) is more important than (2.11). This second proof essentially relies on the following lemma that shows the relations between the solution to the adjoint system (2.7), what is called the resolvent $R(\mu)$, and the spectral projections $P_k$. We prove these relations in the particular case of the heat equation but let us mention that they remain true for a way larger class of systems.

*Remark* 2.3.6. In what follows, the functions we consider are actually complex valued functions. However, splitting into real and imaginary parts the equation, we see that this does not affect the controllability properties.

**Lemma 2.3.7.** *Let*

$$\rho(\Delta) = \mathbb{C} \backslash \left\{ -\widehat{\lambda}_k \right\}_{k \in \mathbb{N}^*},$$

*and let* $R(\mu) : L^2(\Omega) \longrightarrow L^2(\Omega)$ *be the operator defined for every* $\mu \in \rho(\Delta)$ *by*

$$R(\mu) z^1 = \sum_{k=1}^{+\infty} \frac{1}{\mu + \widehat{\lambda}_k} P_k z^1, \quad z^1 \in L^2(\Omega). \tag{2.17}$$

*The set* $\rho(\Delta)$ *is called the resolvent set of* $\Delta$ *and the operator* $R(\mu)$ *is called the resolvent of* $\Delta$*. We have*

*(i)* $\rho(\Delta)$ *is open,* $R(\mu)$ *is a well-defined bounded linear operator for* $\mu \in \rho(\Delta)$ *and for every* $z^1 \in L^2(\Omega)$, $\mu \longmapsto R(\mu) z^1$ *is analytic on* $\rho(\Delta)$.

*(ii) For every* $\mu > -\widehat{\lambda}_1$ *and* $z^1 \in L^2(\Omega)$ *we have*

$$R(\mu) z^1 = \int_0^{+\infty} e^{-\mu t} \tilde{z}(t)\, dt,$$

*where* $\tilde{z}$ *is the solution to* (2.7).

*(iii) For every* $k \in \mathbb{N}^*$ *and* $z^1 \in L^2(\Omega)$ *we have*

$$P_k z^1 = \frac{1}{2\pi i} \int_{C_k} R(\mu) z^1\, d\mu,$$

*where* $C_k = \left\{ \mu \in \mathbb{C}, \left| \mu + \widehat{\lambda}_k \right| \leq r_k \right\}$ *is a positively oriented circle centered in* $-\widehat{\lambda}_k$ *with sufficiently small radius* $r_k > 0$ *so that* $C_k$ *does not contain any other eigenvalues than* $-\widehat{\lambda}_k$ *(this is possible because the eigenvalues* $\left\{ -\widehat{\lambda}_k \right\}_{k \in \mathbb{N}^*}$ *are isolated in* $\mathbb{C}$*).*

*Proof.* As $\widehat{\lambda}_k \to +\infty$, the set $\sigma(\Delta) = \left\{ -\widehat{\lambda}_k \right\}_{k \in \mathbb{N}^*}$ is closed. Indeed, let $v_j \in \sigma(\Delta)$ and $v \in \mathbb{C}$ be such that $v_j \to v$ as $j \to +\infty$ and let us prove that $v \in \sigma(\Delta)$. Note that, necessarily, $v_j, v \in \mathbb{R}$. Since $v_j \in \sigma(\Delta)$, there exists $k_j \in \mathbb{N}^*$ such that $v_j = -\widehat{\lambda}_{k_j}$. Since $v_j \to v$, there exists $J \in \mathbb{N}^*$ such that, for every $j \geq J$, we have $v_j \geq v - 1$. On the other hand, since $-\widehat{\lambda}_k \to -\infty$, there exists $K \in \mathbb{N}^*$ such that, for every $k \geq K$, we have $-\widehat{\lambda}_k < v - 1$. Therefore, for $j \geq J$, we must have $k_j < K$. This shows that $\{v_j\}_{j \in \mathbb{N}^*} = \left\{ -\widehat{\lambda}_{k_1}, \ldots, -\widehat{\lambda}_{k_{J-1}} \right\} \cup \left\{ -\widehat{\lambda}_1, \ldots, -\widehat{\lambda}_{K-1} \right\}$. In particular, $v \in \sigma(\Delta)$.

For $\mu \in \mathbb{C}$, let us introduce the distance from $\mu$ to the set $\sigma(\Delta)$:

$$d(\mu) = d(\mu, \sigma(\Delta)) = \inf_{k \in \mathbb{N}^*} \left| \mu + \widehat{\lambda}_k \right|.$$

Since $\sigma(\Delta)$ is closed we have $d(\mu) = 0$ if, and only if, $\mu \in \sigma(\Delta)$. Let $\mu \in \rho(\Delta)$ be fixed. For $j \in \mathbb{N}^*$, let $S_j = \sum_{k=1}^{j} \frac{1}{\mu + \widehat{\lambda}_k} P_k z^1$. For every $p > q \geq 1$, we have

$$\|S_p - S_q\|^2_{L^2(\Omega)} = \sum_{k=q+1}^{p} \frac{1}{\left|\mu + \widehat{\lambda}_k\right|^2} \left\|P_k z^1\right\|^2_{L^2(\Omega)} \leq \frac{1}{d(\mu)^2} \sum_{k=q+1}^{p} \left\|P_k z^1\right\|^2_{L^2(\Omega)},$$

which proves that $\{S_j\}_{j \in \mathbb{N}^*}$ is a Cauchy sequence in $L^2(\Omega)$. Therefore, $R(\mu)z^1$ is well-defined for every $\mu \in \rho(\Delta)$. Moreover, $R(\mu)$ clearly bounded with

$$\left\|R(\mu)z^1\right\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} \frac{1}{\left|\mu + \widehat{\lambda}_k\right|^2} \left\|P_k z^1\right\|^2_{L^2(\Omega)} \leq \frac{1}{d(\mu)^2} \left\|z^1\right\|^2. \tag{2.18}$$

Let $z^1 \in L^2(\Omega)$ be now fixed. To check that $\mu \longmapsto R(\mu)z^1$ is analytic on $\rho(\Delta)$ we show that it is holomorphic on $\rho(\Delta)$. Let $\mu \in \rho(\Delta)$ be fixed. Let $h \in \rho(\Delta)$ be small enough so that $\mu + h \in \rho(\Delta)$. Formally, we expect to obtain $\frac{d}{dz} R(\mu)z^1 = Q(\mu)$, where

$$Q(\mu) = \sum_{k=1}^{+\infty} \frac{-1}{(\mu + \widehat{\lambda}_k)^2} P_k z^1.$$

The same reasoning as in (2.18) shows that this series is convergent. Let us compute

$$\begin{aligned}\left\| \frac{R(\mu+h)z^1 - R(\mu)z^1}{h} - Q(\mu) \right\|^2_{L^2(\Omega)} &= \sum_{k=1}^{+\infty} \left( \frac{h}{\left|\mu + \widehat{\lambda}_k\right|^2 \left|\mu + h + \widehat{\lambda}_k\right|} \right) \left\|P_k z^1\right\|^2_{L^2(\Omega)} \\ &\leq \frac{h}{d(\mu)^2 d(\mu+h)} \left\|z^1\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $\mu \to d(\mu)$ is continuous on $\mathbb{C}$, this proves that $\frac{d}{dz} R(\mu)z^1 = Q(\mu)$.

To obtain (ii) we multiply the expression of $\tilde{z}$ by $e^{-\mu t}$, integrate over $(0, +\infty)$ and use the fact that $\mu + \widehat{\lambda}_k > 0$ to compute the integral:

$$\int_0^{+\infty} e^{-\mu t} \tilde{z}(t)\, dt = \sum_{k=1}^{+\infty} \left( \int_0^{+\infty} e^{-(\mu + \widehat{\lambda}_k)t}\, dt \right) P_k z^1 = \sum_{k=1}^{+\infty} \frac{-1}{-(\mu + \widehat{\lambda}_k)} P_k z^1 = R(\mu)z^1.$$

Finally, to obtain (iii) we integrate over $C_k$ the expression (2.17) of $R(\mu)z^1$, use Cauchy's integral formula for $j = k$ and Cauchy's integral theorem for $j \neq k$:

$$\frac{1}{2\pi i} \int_{C_k} R(\mu)z^1\, d\mu = \sum_{j=1}^{+\infty} \left( \frac{1}{2\pi i} \int_{C_k} \frac{1}{\mu + \widehat{\lambda}_j}\, d\mu \right) P_j z^1 = \sum_{j=1}^{+\infty} \delta_{kj} P_j z^1 = P_k z^1.$$

All the above inversions between integrals and series can be justified. □

Let us now turn out to the second proof of Theorem 2.3.1.

*Second proof of Theorem 2.3.1.* By Remark 2.3.3, we have to prove that

$$\forall z^1 \in L^2(\Omega), \quad \left(\mathbb{1}_\omega \tilde{z}(t) = 0, \quad \forall t \in [0, +\infty)\right) \Longrightarrow z^1 = 0,$$

where $\tilde{z}$ is the solution to (2.7). Let us introduce

$$N = \left\{z^1 \in L^2(\Omega), \quad \mathbb{1}_\omega \tilde{z}(t) = 0, \quad \forall t \in [0, +\infty)\right\}.$$

We want to establish that $N = \{0\}$. Firstly, let us prove that $N \subset \tilde{N}$ where

$$\tilde{N} = \left\{z^1 \in L^2(\Omega), \quad \mathbb{1}_\omega R(\mu) z^1 = 0, \quad \forall \mu \in \rho(\Delta)\right\},$$

where $\rho(\Delta)$ is the resolvent set of $\Delta$ and $R(\mu)$ is the resolvent of $\Delta$. Let then $z^1 \in N$. Multiplying the identity

$$\mathbb{1}_\omega \tilde{z}(t) = 0$$

by $e^{-\mu t}$ with $\mu > -\widehat{\lambda}_1$ and integrating over $(0, +\infty)$ with respect to $t$ we obtain (see item (ii) of Lemma 2.3.7)

$$\mathbb{1}_\omega R(\mu) z^1 = 0.$$

Since $\mu \mapsto R(\mu) z^1$ is analytic on $\rho(\Delta)$ (see item (i) of Lemma 2.3.7) the previous identity actually holds for every $\mu \in \rho(\Delta)$ and thus $z^1 \in \tilde{N}$. Secondly, let us prove that

$$\tilde{N} \subset \ker P_k, \quad \forall k \in \mathbb{N}^*, \tag{2.19}$$

as it implies that $\tilde{N} = \{0\}$ by (2.13). Let $z^1 \in \tilde{N}$. Then, for every $\mu \in \rho(\Delta)$,

$$\mathbb{1}_\omega R(\mu) z^1 = 0. \tag{2.20}$$

Let $C_k = \left\{\mu \in \mathbb{C}, \left|\mu + \widehat{\lambda}_k\right| \leq r_k\right\}$ be a positively oriented circle centered in $-\widehat{\lambda}_k$ with sufficiently small radius $r_k > 0$ so that $C_k$ does not contain any other eigenvalues than $-\widehat{\lambda}_k$. Integrating (2.20) over $C_k$ gives (see item (iii) of Lemma 2.3.7)

$$\mathbb{1}_\omega P_k z^1 = 0.$$

Since $P_k z^1 \in \ker(-\widehat{\lambda}_k - \Delta)$, (2.15) yields

$$P_k z^1 = 0.$$

Since $k \in \mathbb{N}^*$ was arbitrary, we thus have (2.19).

□

## 2.4 Null-controllability in dimension one: the method of moments

Since we know that the heat equation (2.1) is approximately controllable, let us now investigate the null-controllability of (2.1) (which is a stronger property as observed in Remark 2.2.5). In this section we focus on the one-dimensional problem. Therefore, (2.1) takes the following form:

$$\begin{cases} \partial_t y - \partial_{xx} y = \mathbb{1}_\omega u & \text{in } (0,T)\times(0,L), \\ y(\cdot,0) = y(\cdot,L) = 0 & \text{on } (0,T), \\ y(0,\cdot) = y^0 & \text{in } (0,L), \end{cases} \tag{2.21}$$

where $L > 0$. We recall that in this case, the eigenvalues of $\Delta = \partial_{xx}$ are simple and that they are explicitly given by

$$-\lambda_k = -\frac{k^2\pi^2}{L^2}, \tag{2.22}$$

with the following corresponding normalized eigenfunction:

$$\phi_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right), \quad x \in [0,L].$$

The goal of this section is to introduce the so-called method of moments which is a very powerful and efficient method to solve (mainly) one-dimensional control problems (this method can also be applied to many other types of PDEs). Using this method we will see in particular that:

- (2.21) is null-controllable in time $T$ for every $T > 0$ and every nonempty open subset $\omega \subset (0,L)$.
- The control cost behaves as $Ce^{C/T}$.
- The control can be taken very smooth (see Theorem 2.4.8 below).

More precisely, we will establish the following result:

**Theorem 2.4.1** (Null-controllability). *For every $T > 0$, for every nonempty open subset $\omega \subset (0,L)$, for every $y^0 \in L^2(0,L)$, there exists $u \in L^2(0,T;L^2(0,L))$ such that the corresponding solution $y$ to* (2.21) *satisfies $y(T) = 0$ and, in addition, we have*

$$\|u\|_{L^2(0,T;L^2(0,L))} \leq Ce^{\frac{C}{T}} \left\|y^0\right\|_{L^2(0,L)},$$

*for some $C > 0$ that does not depend on $y^0$ nor on $T$.*

### 2.4.1 Problem of moments

Let us recall the fundamental relation (see (2.5))

$$\left\langle y(T), z^1 \right\rangle_{L^2(0,L)} - \left\langle y^0, z(0) \right\rangle_{L^2(0,L)} = \int_0^T \left\langle u(t), \mathbb{1}_\omega z(t) \right\rangle_{L^2(0,L)} dt, \quad \forall z^1 \in L^2(0,L).$$

Therefore, (2.21) is null-controllable in time $T$ if, and only if, for every $y^0 \in L^2(0,L)$, there exists $u \in L^2(0,T;L^2(0,L))$ such that

$$0 = \left\langle y^0, z(0) \right\rangle_{L^2(0,L)} + \int_0^T \left\langle u(t), \mathbb{1}_\omega z(t) \right\rangle_{L^2(0,L)} dt, \quad \forall z^1 \in L^2(0,L). \tag{2.23}$$

Since

$$\|z(t)\|_{L^2(0,L)} \leq \left\|z^1\right\|_{L^2(0,L)}, \quad \forall t \in [0,T],$$

the right-hand side of (2.23) defines a bounded linear form of $z^1$. Therefore, it is equivalent to test this identity only on a dense subset of $L^2(0,L)$. We choose to do so with the basis $\{\phi_k\}_k$ of eigenfunctions of the Dirichlet Laplacian. Since $z(t) = e^{-\lambda_k(T-t)}\phi_k$ for $z^1 = \phi_k$, (2.23) is then equivalent to

$$0 = \left\langle y^0, e^{-\lambda_k T}\phi_k \right\rangle_{L^2(0,L)} + \int_0^T \left\langle u(t), \mathbb{1}_\omega e^{-\lambda_k(T-t)}\phi_k \right\rangle_{L^2(0,L)} dt, \quad \forall k \in \mathbb{N}^*. \tag{2.24}$$

Let us introduce

$$\alpha_k = -e^{-\lambda_k T}\left\langle y^0, \phi_k \right\rangle_{L^2(0,L)},$$

and

$$\tilde{p}_k(t,x) = \mathbb{1}_\omega(x) e^{-\lambda_k(T-t)}\phi_k(x).$$

Then, (2.24) becomes

$$\langle u, \tilde{p}_k \rangle_{L^2(0,T;L^2(0,L))} = \alpha_k, \quad \forall k \in \mathbb{N}^*. \tag{2.25}$$

Finding $u$ such that (2.25) is a so-called problem of moments.

*Remark* 2.4.2. Observe that it is easy to solve (2.25) if the family $\{\tilde{p}_k\}_{k\in\mathbb{N}^*}$ is orthogonal in $L^2(0,T;L^2(0,L))$. Indeed, in this case, the function $u$ defined by

$$u = \sum_{j=1}^{+\infty} \alpha_j \tilde{p}_j$$

(the series being normally convergent in $L^2(0,T;L^2(0,L))$) provides a solution to (2.25). Unfortunately, the family $\{\tilde{p}_k\}_{k\in\mathbb{N}^*}$ is never orthogonal in $L^2(0,T;L^2(0,L))$, unless $\omega = (0,L)$ (but this case is not interesting, see Remark 2.2.3).

The previous remark leads us to introduce the notion of biorthogonal family, which is the core of the method of moments.

**Definition 2.4.3** (Biorthogonal family)**.** Let $H$ be a Hilbert space. We say that two families $\{q_j\}_{j\in\mathbb{N}^*}$ and $\{p_k\}_{k\in\mathbb{N}^*}$ are biorthogonal in $H$ if

$$\langle q_j, p_k\rangle_H = \delta_{jk}, \quad \forall j,k \in \mathbb{N}^*.$$

The goal is then to construct a family $\{\tilde{q}_j\}_{j\in\mathbb{N}^*}$ biorthogonal to $\{\tilde{p}_k\}_{k\in\mathbb{N}^*}$ in $L^2(0,T;L^2(0,L))$ and such that the series

$$u = \sum_{j=1}^{+\infty} \alpha_j \tilde{q}_j \tag{2.26}$$

converges in $L^2(0,T;L^2(0,L))$. Since $\tilde{p}_k$ is a function with separated variables we readily see that, if there exists a family $\{q_j\}_{j\in\mathbb{N}^*}$ biorthogonal to $\left\{e^{-\lambda_k(T-\cdot)}\right\}_{k\in\mathbb{N}^*}$ in $L^2(0,T)$, then the family defined by

$$\tilde{q}_j(t,x) = q_j(t)\frac{\phi_j(x)}{\|\mathbb{1}_\omega\phi_j\|^2_{L^2(0,L)}}, \tag{2.27}$$

(provided that $\mathbb{1}_\omega\phi_j \neq 0$) is biorthogonal to $\{\tilde{p}_k\}_{k\in\mathbb{N}^*}$ in $L^2(0,T;L^2(0,L))$. Moreover, to estimate the norm of $\tilde{q}_j$ in view of the convergence of the series in (2.26), we see that it is enough to bound from below the norm of the observations of the eigenfunctions and to bound from above the norm of the elements of the biorthogonal family. Let us make all of this precise with the two following results:

**Lemma 2.4.4.** *There exists $C>0$ such that*

$$\|\mathbb{1}_\omega\phi_j\|_{L^2(0,L)} \geq C, \quad \forall j \in \mathbb{N}^*.$$

*Proof.* First of all recall that $\phi_j(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{j\pi}{L}x\right)$ and observe that it is enough to prove the estimate for $j$ large enough since $\phi_j$ can not be identically zero on $\omega$ by analyticity. Since $\omega$ is a nonempty open subset, it contains an interval $(a,b)\subset\omega$, where $0<a<b<L$. A computation shows that

$$\int_a^b \left|\sin\left(\frac{j\pi}{L}x\right)\right|^2 dx = \frac{b-a}{2} - \frac{\sin\left(2\frac{j\pi}{L}b\right) - \sin\left(2\frac{j\pi}{L}a\right)}{4\frac{j\pi}{L}}.$$

Since the second term in the right-hand side goes to 0 as $j\to+\infty$ we get the conclusion. □

**Theorem 2.4.5** (Existence of a biorthogonal family)**.** *Let $\{\lambda_k\}_{k\in\mathbb{N}^*}$ be given by* (2.22)*. For every $T>0$, there exists a family $\{q_j\}_{j\in\mathbb{N}^*}\subset C^\infty([0,T])$ such that:*

*(i) $\{q_j\}_{j\in\mathbb{N}^*}$ biorthogonal to $\left\{e^{-\lambda_k(T-\cdot)}\right\}_{k\in\mathbb{N}^*}$ in $L^2(0,T)$.*

*(ii) For every $j \in \mathbb{N}^*$ and $r \in \mathbb{N}$, we have*

$$\left| q_j^{(r)}(t) \right| \leq C e^{\frac{C}{T} + \frac{\lambda_j T}{2}}, \quad \forall t \in [0, T], \tag{2.28}$$

*for some $C > 0$ that does not depend on $j$ nor on $t, T$.*

*Proof.* The proof of Theorem 2.4.5 is the purpose of Section 2.4.2 below. □

Let us now go back to the convergence of the series in (2.26). Actually, we shall prove that this series converges in $C^0([0,T]; L^2(0,L))$ (and thus in $L^2(0,T; L^2(0,L))$). More precisely, for every $t \in [0,T]$, we set

$$u(t) = \sum_{j=1}^{+\infty} \alpha_j \tilde{q}_j(t), \tag{2.29}$$

where $\tilde{q}_j$ is defined by (2.27). Let us now show that, for every $t \in [0,T]$, this series is normally convergent in $L^2(0,L)$ and that we have the estimate

$$\|u(t)\|_{L^2(0,L)} \leq C e^{\frac{C}{T}} \left\| y^0 \right\|_{L^2(0,L)},$$

for some $C > 0$ that does not depend on $j$ nor on $t, T$. Clearly,

$$|\alpha_j| \leq e^{-\lambda_j T} \left| \left\langle y^0, \phi_j \right\rangle_{L^2(0,L)} \right|.$$

On the other hand, using Lemma 2.4.4 and Theorem 2.4.5 we see that

$$\|\tilde{q}_j(t)\|_{L^2(0,L)} \leq C e^{\frac{C}{T} + \frac{\lambda_j T}{2}}, \quad \forall t \in [0,T].$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \sum_{j=1}^{+\infty} \|\alpha_j \tilde{q}_j(t)\|_{L^2(0,L)} &\leq C e^{\frac{C}{T}} \sum_{j=1}^{+\infty} e^{-\frac{\lambda_j T}{2}} \left| \left\langle y^0, \phi_j \right\rangle_{L^2(0,L)} \right| \\ &\leq C e^{\frac{C}{T}} \left( \sqrt{\sum_{j=1}^{+\infty} e^{-\lambda_j T}} \right) \left\| y^0 \right\|_{L^2(0,L)}. \end{aligned}$$

Next, observe that

$$\sum_{j=1}^{+\infty} e^{-\lambda_j T} = \frac{1}{T} \sum_{j=1}^{+\infty} \left( \lambda_j T e^{-\lambda_j T} \right) \frac{1}{\lambda_j} \leq \frac{1}{T} \left( \sup_{x>0} x e^{-x} \right) \sum_{j=1}^{+\infty} \frac{1}{\lambda_j}.$$

Since

$$\sum_{j=1}^{+\infty} \frac{1}{\lambda_j} < +\infty,$$

this concludes the proof of Theorem 2.4.1.

*Remark* 2.4.6. We easily see from the expression (2.29) and the properties of $\{\tilde{q}_j\}_{j\in\mathbb{N}^*}$ that, in fact, $u \in C^\infty([0,T]; L^2(0,L))$ with, for every $r \in \mathbb{N}$,

$$\left\| u^{(r)}(t) \right\|_{L^2(0,L)} \leq C e^{\frac{C}{T}} \left\| y^0 \right\|_{L^2(0,L)}, \quad \forall t \in [0,T],$$

for some $C > 0$ that does not depend on $y^0$ nor on $t, T$.

*Remark* 2.4.7. The control $u$ in Theorem 2.4.1 can even be chosen space-independent. Indeed, instead of choosing the family $\{\tilde{q}_j\}_{j\in\mathbb{N}^*}$ defined by (2.27), we can take the space-independent family

$$\tilde{q}_j(t) = q_j(t) \frac{1}{\int_a^b \phi_j(x)\, dx},$$

where $(a,b) \subset \omega$ is such that

$$\left| \int_a^b \phi_j(x)\, dx \right| \geq \frac{C}{\lambda_j^p}, \quad \forall j \in \mathbb{N}^*,$$

for some $C > 0$ and $p \in \mathbb{N}$ that do not depend on $j$. The existence of such an interval can be established using Liouville's theorem on Diophantine approximation.

In view of the two previous remarks, we have actually the following result:

**Theorem 2.4.8.** *For every $T > 0$, for every nonempty open subset $\omega \subset (0,L)$, for every $y^0 \in L^2(0,L)$, there exists $u \in C^\infty([0,T])$ such that the corresponding solution $y$ to* (2.21) *satisfies $y(T) = 0$ and, in addition, we have, for every $r \in \mathbb{N}$,*

$$\left| u^{(r)}(t) \right| \leq C e^{\frac{C}{T}} \left\| y^0 \right\|_{L^2(0,L)}, \quad \forall t \in [0,T],$$

*for some $C > 0$ that does not depend on $y^0$ nor on $t, T$.*

### 2.4.2 Construction of a biorthogonal family to the exponentials

This section is devoted to the proof of Theorem 2.4.5. We recall that we want to prove that there exists a family $\{q_j\}_{j\in\mathbb{N}^*}$ such that

$$\int_0^T q_j(t) e^{-\lambda_k t}\, dt = \delta_{jk}, \quad \forall j,k \in \mathbb{N}^*,$$

(we performed the change of variables $T - t \mapsto t$ for the sake of simplicity) where

$$\lambda_k = \frac{k^2 \pi^2}{L^2}.$$

*Remark* 2.4.9. As we have already seen at the end of the proof of Theorem 2.4.1, note that

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} < +\infty. \tag{2.30}$$

This condition is mandatory to prove the existence of a biorthogonal family to $\left\{e^{-\lambda_k \cdot}\right\}_{k\in\mathbb{N}^*}$ in $L^2(0,T)$. Indeed, a consequence of the Müntz-Szasz theorem is that, if this series diverges, then $\left\{e^{-\lambda_k \cdot}\right\}_{k\in\mathbb{N}^*}$ is dense in $L^2(0,T)$. In particular, for every $j \in \mathbb{N}^*$, the family $\left\{e^{-\lambda_k \cdot}\right\}_{\substack{k\in\mathbb{N}^* \\ k\neq j}}$ is still dense in $L^2(0,T)$. It follows that if a function $q$ is orthogonal to $e^{-\lambda_k \cdot}$ for every $k \neq j$, then it is necessarily zero and therefore it can not satisfy the remaining condition that $\int_0^T q(t)e^{-\lambda_j t}\,dt = 1$.

*Remark* 2.4.10. Note that, once the family $\{q_j\}_{j\in\mathbb{N}^*}$ is constructed, the family $\{\tilde{q}_j\}_{j\in\mathbb{N}^*}$ defined by (2.27) is a biorthogonal to $\{\tilde{p}_k\}_{k\in\mathbb{N}^*}$, whatever the space dimension is. Thus, we see that the necessary condition (2.30) is the main obstruction to make the method of moments works in higher space dimension since, by the Weyl's law, the eigenvalues of the Dirichlet Laplacian on a smooth bounded open subset of $\mathbb{R}^N$ satisfy $\lambda_k \sim Ck^{\frac{2}{N}}$ as $k \to +\infty$.

The construction of the biorthogonal family presented here relies on the celebrated Paley-Wiener theorem that gives (necessary and) sufficient conditions for a function to be the Fourier transform of a complex function with compact support (for a proof, see e.g. [You80, Theorem 4.18]). Let us point out that, from now on, all the functions we consider are actually complex valued functions.

**Theorem 2.4.11** (Paley-Wiener)**.** *Let $\hat{g} : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. Assume that $\hat{g}$ is of exponential type $\tau > 0$, that is, there exist $C > 0$ such that*

$$|\hat{g}(z)| \leq Ce^{\tau|z|}, \quad \forall z \in \mathbb{C}.$$

*Assume in addition that*

$$\int_{-\infty}^{+\infty} |\hat{g}(x)|^2\,dx < +\infty. \tag{2.31}$$

*Then, there exists $g \in L^2(-\tau,\tau)$ such that*

$$\hat{g}(z) = \int_{-\tau}^{\tau} g(t)e^{-izt}\,dt, \quad \forall z \in \mathbb{C}.$$

The goal is then to find a function $\hat{g}_j$ such that

$$\hat{g}_j(-i\lambda_k) = \delta_{jk}, \quad \forall j,k \in \mathbb{N}^*,$$

and such that all the assumptions of Theorem 2.4.11 are satisfied. The first step is thus to construct a function which has $-i\lambda_k$ as zeros. A natural candidate is the infinite product

$$p(z) = \prod_{k=1}^{+\infty}\left(1 + \frac{z}{i\lambda_k}\right), \quad z \in \mathbb{C}.$$

Note that this infinite product is absolutely convergent thanks to (2.30). Next, observe that each $-i\lambda_k$ is actually a simple zero of $p$. This shows that the function

$$p_j(z) = \prod_{\substack{k=1 \\ k\neq j}}^{+\infty} \left(1 + \frac{z}{i\lambda_k}\right), \quad z \in \mathbb{C},\, j \in \mathbb{N}^*,$$

satisfies $p_j(-i\lambda_j) \neq 0$. Therefore, we can introduce

$$F_j(z) = \frac{p_j(z)}{p_j(-i\lambda_j)}, \quad z \in \mathbb{C}. \tag{2.32}$$

Finally, observe that $p_j$ is an entire function since all its factors are entire functions.

**Lemma 2.4.12.** *The entire function $F_j$ defined by* (2.32) *has the following properties:*

*(i)* $F_j(-i\lambda_k) = \delta_{jk}$ *for every* $k \in \mathbb{N}^*$.

*(ii)* $|F_j(z)| \leq 2e^{L\sqrt{|z|}}$ *for every* $z \in \mathbb{C}$.

*(iii)* $|F_j(x)| \leq 2e^{L\sqrt{\frac{|x|}{2}}}$ *for every* $x \in \mathbb{R}$.

*Proof.* The proof of item (i) is clear by construction. Let us check (ii). First of all, let us recall Euler's infinite product for the sine function:

$$\sin(z) = z\prod_{k=1}^{+\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right), \quad \forall z \in \mathbb{C}.$$

Let then *sinc* be the cardinal sine function:

$$sinc\,(z) = \frac{sin(z)}{z}, \quad \forall z \in \mathbb{C}\backslash\{0\}, \qquad sinc\,(0) = 1.$$

Since $\lambda_k = \frac{k^2\pi^2}{L^2}$, we see that

$$p(z) = sinc\,\left(L\sqrt{iz}\right), \quad \forall z \in \mathbb{C}, \tag{2.33}$$

where $\sqrt{z}$ denotes the principal value of the square root of $z \in \mathbb{C}$. From the identity

$$sinc\,(z) = \frac{1}{2}\int_{-1}^{1} e^{izt}\,dt, \quad \forall z \in \mathbb{C},$$

it follows that

$$|sinc\,(z)| \leq e^{|\mathrm{Im}\, z|} \leq e^{|z|}, \quad \forall z \in \mathbb{C}. \tag{2.34}$$

This establishes that

$$|p(z)| \leq e^{L\sqrt{|z|}}, \quad \forall z \in \mathbb{C}. \tag{2.35}$$

Next, observe that, for every $z \in \mathbb{C}$, we have

$$|p_j(z)| \leq \prod_{\substack{k=1\\k\neq j}}^{+\infty} \left(1 + \frac{|z|}{\lambda_k}\right) \leq \prod_{k=1}^{+\infty} \left(1 + \frac{|z|}{\lambda_k}\right) = p(i\,|z|).$$

Therefore, (2.35) yields

$$|p_j(z)| \leq e^{L\sqrt{|z|}}, \quad \forall z \in \mathbb{C}. \tag{2.36}$$

Taking the derivative at $z = -i\lambda_j$ of the identity

$$p(z) = \left(1 + \frac{z}{i\lambda_j}\right) p_j(z),$$

we obtain the relation

$$p_j(-i\lambda_j) = i\lambda_j p'(-i\lambda_j).$$

Let us compute $p'(-i\lambda_j)$. First, we recall that the function $z \mapsto \sqrt{z}$ is derivable on $\mathbb{C}\setminus\{x \in \mathbb{R},\, x \leq 0\}$ with derivative $\frac{1}{2\sqrt{z}}$. Since $\lambda_j \notin \{x \in \mathbb{R},\, x \leq 0\}$ we can use the chain rule in the expression (2.33) to obtain that

$$p'(-i\lambda_j) = \frac{(-1)^j}{2\lambda_j} i.$$

This shows that

$$|p_j(-i\lambda_j)| = \frac{1}{2}. \tag{2.37}$$

Combining (2.37) with (2.36) we obtain (ii). To prove the remaining property (iii), we go back to the first inequality in (2.34) for $z = L\sqrt{ix}$. Since $\operatorname{Im}\sqrt{ix} = \frac{\sqrt{|x|}}{\sqrt{2}}$, this gives

$$|p(x)| \leq e^{L\sqrt{\frac{|x|}{2}}}.$$

To conclude, note that, for every $x \in \mathbb{R}$,

$$|p_j(x)| \leq |p(x)|,$$

since $\left|1 + \frac{x}{i\lambda_j}\right| \geq 1$. □

The two first items in Lemma 2.4.12 are good. However, because of third one, we see that Theorem 2.4.11 can not be applied to the function $F_j$ itself (it does not satisfy (2.31)). Therefore, we need to compensate this bad behavior along the real axis. To this end, we will multiply $F_j$ by a function that behaves as least like $e^{-\frac{L}{\sqrt{2}}\sqrt{|x|}}$ as $x \to \pm\infty$. Such a function is called multiplier.

**Lemma 2.4.13** (Multiplier). *There exists $C>0$ such that, for every $\beta,\nu>0$, there exists an entire function $M_{\beta,\nu}:\mathbb{C}\longrightarrow\mathbb{C}$ such that:*

*(i)* $|M_{\beta,\nu}(z)|\leq e^{\beta|z|}$ *for every* $z\in\mathbb{C}$.

*(ii)* $|M_{\beta,\nu}(x)|\leq Ce^{C\nu+\sqrt{\beta\nu}}e^{-\sqrt{\frac{\beta\nu|x|}{2}}}$ *for every* $x\in\mathbb{R}$.

*(iii)* $M_{\beta,\nu}(ix)\geq Ce^{-C\nu}$ *for every* $x\in\mathbb{R}$.

*Idea of the proof.* The function $M_{\beta,\nu}$ is

$$M_{\beta,\nu}(z)=\frac{1}{\|\sigma_\nu\|_{L^1(\mathbb{R})}}\int_{-1}^{1}\sigma_\nu(t)e^{-it\beta z}\,dt,\quad z\in\mathbb{C},$$

where the function $\sigma_\nu\in C_c^\infty(\mathbb{R})$ is given by

$$\sigma_\nu(t)=\begin{cases}-\dfrac{\nu}{1-t^2} & \text{if } t\in(-1,1),\\ 0 & \text{otherwise.}\end{cases}$$

For the proof of the estimates, we refer to [TT07, Lemma 4.3]. □

We are now in position to prove Theorem 2.4.5.

*Proof of Theorem 2.4.5.* Let $j\in\mathbb{N}^*$ be fixed. We set

$$\hat{g}_j(z)=F_j(z)\frac{M_{\beta,\nu}(z)}{M_{\beta,\nu}(-i\lambda_j)},\quad z\in\mathbb{C},$$

with

$$\beta=\varepsilon\frac{T}{2},\qquad 0<\varepsilon<1,$$

and

$$\nu=\frac{8L^2}{\varepsilon T},$$

so that

$$L-\sqrt{\beta\nu}\leq -L. \tag{2.38}$$

It is clear by construction that $\hat{g}_j$ is an entire function and that we have

$$\hat{g}_j(-i\lambda_k)=\delta_{jk},\quad \forall k\in\mathbb{N}^*.$$

Using the estimates of $F_j$ and $M_{\beta,\nu}$, we have

$$|\hat{g}_j(z)|\leq\frac{2e^{C\nu}}{C}e^{L\sqrt{|z|}+\beta|z|},\quad \forall z\in\mathbb{C},$$

where $C>0$ is the constant provided by Lemma 2.4.13. Combined with Young's inequality

$$L\sqrt{|z|} \leq \left(\frac{T}{2}-\beta\right)|z| + \frac{L^2}{4\left(\frac{T}{2}-\beta\right)},$$

this shows that $\hat{g}_j$ is an entire function of exponential type $T/2$. Finally, using the estimates of $F_j$ and $M_{\beta,\nu}$ and using (2.38), we obtain

$$|\hat{g}_j(x)| \leq 2e^{2C\nu+\sqrt{\beta\nu}}e^{-L\sqrt{\frac{|x|}{2}}}, \quad \forall x \in \mathbb{R}. \tag{2.39}$$

Therefore, $\hat{g}_j \in L^2(\mathbb{R})$. By Theorem 2.4.11, there exists $g_j \in L^2(-T/2, T/2)$ such that

$$\hat{g}_j(z) = \int_{-T/2}^{T/2} g_j(t)e^{-izt}\,dt, \quad \forall z \in \mathbb{C}.$$

Let $q_j$ be defined by

$$q_j(t) = e^{\frac{\lambda_j T}{2}} \operatorname{Re} g_j\left(t-\frac{T}{2}\right), \quad \text{a.e. } t \in (0,T). \tag{2.40}$$

Then, $q_j \in L^2(0,T)$ and the family $\{q_j\}_{j\in\mathbb{N}^*}$ is biorthogonal to $\left\{e^{-\lambda_k \cdot}\right\}_{k\in\mathbb{N}^*}$ in $L^2(0,T)$.

Let us now establish that $q_j \in C^\infty([0,T])$ with the desired estimate (2.28). Since $g_j \in L^2(-T/2,T/2)$, in particular $g_j \in L^1(\mathbb{R})$. Thus, $\hat{g}_j$ is actually the Fourier transform (in $L^1(\mathbb{R})$) of $g_j$. In addition, note that from (2.39) we have $\hat{g}_j \in L^1(\mathbb{R})$. Therefore, the Fourier inversion theorem gives

$$g_j(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}_j(x)e^{itx}\,dx, \quad \forall t \in \mathbb{R}.$$

From (2.39) and the definition of $\beta,\nu$, we also see that, for every $r\in\mathbb{N}$, the map $x \mapsto x^r\hat{g}_j(x)$ belongs to $L^1(\mathbb{R})$ with

$$\|x^r\hat{g}_j\|_{L^1(\mathbb{R})} \leq Ke^{\frac{K}{T}},$$

for some $K>0$ that does not depend on $j$ nor on $T$. It follows that, for every $r \in \mathbb{N}$, we have $g_j \in C^r(\mathbb{R})$ with, for every $t\in\mathbb{R}$,

$$\left|g_j^{(r)}(t)\right| \leq \frac{1}{2\pi}\|x^r\hat{g}_j\|_{L^1(\mathbb{R})} \leq \frac{K}{2\pi}e^{\frac{K}{T}}.$$

Coming back to the definition (2.40), we see that this establishes (2.28). □

## 2.5 Null-controllability in any dimension: the Lebeau-Robbiano method

The goal of this section is to generalize Theorem 2.4.1 to any space dimension. More precisely, we will prove the following:

**Theorem 2.5.1** (Null-controllability). *For every $T > 0$, for every nonempty open subset $\omega \subset \Omega$, for every $y^0 \in L^2(\Omega)$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution $y$ to* (2.1) *satisfies $y(T) = 0$ and, in addition, we have*

$$\|u\|_{L^2(0,T;L^2(\Omega))} \leq Ce^{\frac{C}{T}} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.41}$$

*for some $C > 0$ that does not depend on $y^0$ nor on $T$.*

*Remark* 2.5.2. Using the decreasing of the null-control cost, we see that it is enough to establish (2.41) only for small times $T$.

### 2.5.1 Partial controllability

The proof of Theorem 2.5.1 relies on the following fundamental inequality. This inequality can be obtained by means of global elliptic Carleman estimates (the proof is admitted here).

**Theorem 2.5.3** (Lebeau-Robbiano inequality). *There exists $C > 0$ such that, for every $\mu > 0$, for every $(a_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}$, we have*

$$\left\| \sum_{k:\, \lambda_k \leq \mu} a_k \phi_k \right\|^2_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}} \left\| \sum_{k:\, \lambda_k \leq \mu} a_k \mathbb{1}_\omega \phi_k \right\|^2_{L^2(\Omega)}, \tag{2.42}$$

*(with the convention that the sum is equal to zero if $\mu < \lambda_1$).*

Let us introduce the subspaces

$$E_\mu = \operatorname{span}\left\{\phi_k, \quad k:\, \lambda_k \leq \mu\right\},$$

and let $P_{E_\mu} : L^2(\Omega) \longrightarrow L^2(\Omega)$ be the orthogonal projection on $E_\mu$. Therefore,

$$P_{E_\mu} z = \sum_{k:\, \lambda_k \leq \mu} \langle z, \phi_k \rangle_{L^2(\Omega)} \phi_k, \quad z \in L^2(\Omega).$$

Then, (2.42) can be restated as

$$\left\|P_{E_\mu} z\right\|^2_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}} \left\|\mathbb{1}_\omega P_{E_\mu} z\right\|^2_{L^2(\Omega)}, \quad \forall z \in L^2(\Omega). \tag{2.43}$$

**Proposition 2.5.4** (Partial observability). *There exists $C > 0$ such that, for every $\mu > 0$ and $T > 0$, we have*

$$\|z(0)\|_{L^2(\Omega)}^2 \leq \frac{C}{T} e^{C\sqrt{\mu}} \int_0^T \|\mathbb{1}_\omega z(t)\|_{L^2(\Omega)}^2 \, dt, \quad \forall z^1 \in E_\mu, \tag{2.44}$$

*where $z$ is the solution to the adjoint system (2.6).*

*Proof.* Since $E_\mu$ is stable by the Dirichlet Laplacian, for $z^1 \in E_\mu$ we have $z(t) \in E_\mu$ for every $t \in [0,T]$. Applying the Lebeau-Robbiano inequality (2.43) to $z = z(t)$ we obtain

$$\|z(t)\|_{L^2(\Omega)}^2 \leq C e^{C\sqrt{\mu}} \|\mathbb{1}_\omega z(t)\|_{L^2(\Omega)}^2, \quad \forall t \in [0,T].$$

Using the dissipation property

$$\|z(0)\|_{L^2(\Omega)}^2 \leq e^{-2\lambda_1 t} \|z(t)\|_{L^2(\Omega)}^2, \quad \forall t \in [0,T],$$

and integrating over $(0,T)$ gives (2.44). □

By duality we can obtain the following:

**Proposition 2.5.5** (Partial controllability). *Let $\mu > 0$ and $T > 0$. For every $y^0 \in L^2(\Omega)$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution $y$ to (2.1) satisfies*

$$P_{E_\mu} y(T) = 0,$$

*and*

$$\|u\|_{L^2(0,T;L^2(\Omega))} \leq \frac{C}{\sqrt{T}} e^{C\sqrt{\mu}} \left\|y^0\right\|_{L^2(\Omega)},$$

*for some $C > 0$ that does not depend on $y^0$ nor on $\mu, T$.*

Let us now use the natural dissipation of the equation (2.1).

**Proposition 2.5.6.** *Let $\mu > 0$ and $T > 0$. For every $y^0 \in L^2(\Omega)$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution $y$ to (2.1) satisfies*

$$\|y(T)\|_{L^2(\Omega)} \leq C e^{C\sqrt{\mu} - \frac{T\mu}{2}} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.45}$$

*and*

$$\|u\|_{L^2(0,T;L^2(\Omega))} \leq \frac{C}{\sqrt{T}} e^{C\sqrt{\mu}} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.46}$$

*for some $C > 0$ that does not depend on $y^0$ nor on $\mu, T$.*

*Proof.* We start by applying Proposition 2.5.5 on the time interval $(0,T/2)$. This gives the existence of $\tilde{u} \in L^2(0,T/2;L^2(\Omega))$ such that the corresponding solution $\tilde{y}$ satisfies

$$P_{E_\mu}\tilde{y}(T/2) = 0,$$

and, in addition, we have

$$\|\tilde{u}\|_{L^2(0,T/2;L^2(\Omega))} \leq \frac{C}{\sqrt{T/2}} e^{C\sqrt{\mu}} \left\|y^0\right\|_{L^2(\Omega)}. \tag{2.47}$$

We then define

$$u(t) = \begin{cases} \tilde{u}(t) & \text{if } t \in (0,T/2), \\ 0 & \text{if } t \in (T/2,T). \end{cases}$$

Clearly, $u$ satisfies (2.46). Let $y$ be the corresponding solution on $(0,T)$. Clearly, $y = \tilde{y}$ on $[0,T/2]$. Combining (2.4) with (2.47), we have

$$\begin{aligned} \|y(T/2)\|_{L^2(\Omega)} &\leq \left(1 + Ce^{C\sqrt{\mu}}\right) \left\|y^0\right\|_{L^2(\Omega)} \\ &\leq (1+C)e^{C\sqrt{\mu}} \left\|y^0\right\|_{L^2(\Omega)}. \end{aligned} \tag{2.48}$$

Finally, on $(T/2,T)$, since $u = 0$ and the "initial data" $y(T/2)$ satisfies $P_{E_\mu}y(T/2) = 0$, we have

$$\|y(T)\|^2_{L^2(\Omega)} = \sum_{k:\,\lambda_k > \mu} e^{-\lambda_k T} \left|\langle y(T/2), \phi_k\rangle_{L^2(\Omega)}\right|^2,$$

which gives the following dissipation property

$$\|y(T)\|_{L^2(\Omega)} \leq e^{-(T/2)\mu} \|y(T/2)\|_{L^2(\Omega)}. \tag{2.49}$$

Combining (2.49) with (2.48) we obtain (2.45). □

In the next section, we will need to consider the equation (2.1) on intervals of the form $(t_0, t_0+T)$ with $t_0 \geq 0$ not necessarily equal to 0. Therefore, we restate below Proposition 2.5.6 in this framework (the proof is obvious by performing the change of variables $t \mapsto t + t_0$).

**Proposition 2.5.7.** *Let $\mu > 0$ and $t_0 \geq 0$, $T > 0$. For every $y^0 \in L^2(\Omega)$, there exists $u \in L^2(t_0, t_0+T; L^2(\Omega))$ such that the corresponding solution $y$ to*

$$\begin{cases} \partial_t y - \Delta y = \mathbb{1}_\omega u & \text{in } (t_0, t_0+T) \times \Omega, \\ y = 0 & \text{on } (t_0, t_0+T) \times \partial\Omega, \\ y(t_0) = y^0 & \text{in } \Omega, \end{cases}$$

*satisfies*

$$\|y(t_0+T)\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu} - \frac{T\mu}{2}} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.50}$$

*and, in addition, we have*

$$\|u\|_{L^2(t_0,t_0+T;L^2(\Omega))} \leq \frac{C}{\sqrt{T}} e^{C\sqrt{\mu}} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.51}$$

*for some* $C > 0$ *that does not depend on* $y^0$ *nor on* $\mu, t_0, T$.

### 2.5.2 Time-splitting procedure

Let $C > 0$ be the constant provided by Proposition 2.5.7. We emphasize that, in what follows, $C$ exclusively refers to this constant and does not change from line to line.

We split the time interval $(0, T)$ into smaller intervals of sizes $T_k$, $k \in \mathbb{N}^*$, with

$$\sum_{k=1}^{+\infty} T_k = T,$$

and we successively apply a partial control provided by Proposition 2.5.7 with a cut frequency $\mu_k$ that goes appropriately to $+\infty$ as $k \to +\infty$. More precisely, we define

$$T_k = \frac{T}{2^k}, \quad \mu_k = \alpha(2^k)^2,$$

where $\alpha > 0$ will be chosen below (depending only on $T$ and $C$). On the time interval $(0, T_1)$ we apply Proposition 2.5.7 with $\mu = \mu_1$, $t_0 = 0$, $T = T_1$ and initial data $y^0$, which gives the existence of $u_1 \in L^2(0, T_1; L^2(\Omega))$ such that the corresponding solution $y_1$ satisfies

$$\|y_1(T_1)\|_{L^2(\Omega)} \leq C e^{C\sqrt{\mu_1} - \frac{T_1\mu_1}{2}} \left\|y^0\right\|_{L^2(\Omega)},$$

and, in addition, we have

$$\|u_1\|_{L^2(0,T_1;L^2(\Omega))} \leq \frac{C}{\sqrt{T_1}} e^{C\sqrt{\mu_1}} \left\|y^0\right\|_{L^2(\Omega)}.$$

Next, on the time interval $(T_1, T_1 + T_2)$ we apply once again Proposition 2.5.7, this time with $\mu = \mu_2$, $t_0 = T_1$, $T = T_2$ and initial data $y^0 = y_1(T_1)$, which gives the existence of $u_2 \in L^2(T_1, T_1 + T_2; L^2(\Omega))$ such that the corresponding solution $y_2$ satisfies

$$\begin{aligned}\|y_2(T_1 + T_2)\|_{L^2(\Omega)} &\leq C e^{C\sqrt{\mu_2} - \frac{T_2\mu_2}{2}} \|y_1(T_1)\|_{L^2(\Omega)} \\ &\leq C^2 e^{C(\sqrt{\mu_1}+\sqrt{\mu_2}) - \frac{T_1\mu_1+T_2\mu_2}{2}} \left\|y^0\right\|_{L^2(\Omega)},\end{aligned}$$

and, in addition, we have

$$\begin{aligned}\|u_2\|_{L^2(T_1,T_1+T_2;L^2(\Omega))} &\leq \frac{C}{\sqrt{T_2}} e^{C\sqrt{\mu_2}} \|y_1(T_1)\|_{L^2(\Omega)} \\ &\leq \frac{C^2}{\sqrt{T_2}} e^{C(\sqrt{\mu_1}+\sqrt{\mu_2}) - \frac{T_1\mu_1}{2}} \left\|y^0\right\|_{L^2(\Omega)}.\end{aligned}$$

Introducing

$$\tau_j = \sum_{k=1}^{j} T_k, \quad \tau_0 = 0, \quad j \in \mathbb{N}^*,$$

we obtain by induction that, for every $j \in \mathbb{N}^*$, there exists $u_j \in L^2(\tau_{j-1}, \tau_j; L^2(\Omega))$ such that the corresponding solution $y_j$ satisfies

$$\|y_j(\tau_j)\|_{L^2(\Omega)} \leq C^j e^{C \sum_{k=1}^{j} \sqrt{\mu_k} - \sum_{k=1}^{j} \frac{T_k \mu_k}{2}} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.52}$$

and, in addition, we have

$$\|u_j\|_{L^2(\tau_{j-1},\tau_j;L^2(\Omega))} \leq \frac{C^j}{\sqrt{T_j}} e^{C \sum_{k=1}^{j} \sqrt{\mu_k} - \sum_{k=1}^{j-1} \frac{T_k \mu_k}{2}} \left\|y^0\right\|_{L^2(\Omega)}. \tag{2.53}$$

By construction, we have

$$\begin{aligned} C \sum_{k=1}^{j} \sqrt{\mu_k} - \sum_{k=1}^{j} \frac{T_k \mu_k}{2} &= C\sqrt{\alpha} \sum_{k=1}^{j} 2^k - \frac{\alpha T}{2} \sum_{k=1}^{j} 2^k \\ &= \left(C\sqrt{\alpha} - \frac{\alpha T}{2}\right)(2^{j+1} - 2) \\ &= -\beta 2^j + \beta, \end{aligned}$$

where we set $\beta = -2\left(C\sqrt{\alpha} - \frac{\alpha T}{2}\right)$. On the other hand,

$$C \sum_{k=1}^{j} \sqrt{\mu_k} - \sum_{k=1}^{j-1} \frac{T_k \mu_k}{2} = -\beta 2^j + \beta + \frac{T_j \mu_j}{2} = -\gamma 2^j + \beta,$$

where we set $\gamma = -\left(2C\sqrt{\alpha} - \frac{\alpha T}{2}\right)$. Thus, coming back to (2.52) and (2.53) we obtain

$$\|y_j(\tau_j)\|_{L^2(\Omega)} \leq e^{\beta} C^j e^{-\beta 2^j} \left\|y^0\right\|_{L^2(\Omega)}, \tag{2.54}$$

and

$$\|u_j\|_{L^2(\tau_{j-1},\tau_j;L^2(\Omega))} \leq e^{\beta} \frac{\sqrt{2}^j}{\sqrt{T}} C^j e^{-\gamma 2^j} \left\|y^0\right\|_{L^2(\Omega)}.$$

Let us now fix $\alpha > 0$ large enough (and depending only on $T$ and $C$) so that $\beta > 0$ and $\gamma > 0$. More precisely, we set

$$\alpha = \frac{36C^2}{T^2}.$$

Thus,

$$\beta = \frac{24C^2}{T}, \quad \gamma = \frac{6C^2}{T}. \tag{2.55}$$

Let now $p \in \mathbb{N}$ be large enough (and depending only on $C$) so that

$$\frac{\sqrt{2}C}{2^p} < 1. \tag{2.56}$$

Next, observe that, since $\gamma > 0$, we have

$$\begin{aligned}(\sqrt{2}C)^j e^{-\gamma 2^j} &= \frac{1}{\gamma^p}\left(\frac{\sqrt{2}C}{2^p}\right)^j \left(\gamma 2^j\right)^p e^{-\gamma 2^j} \\ &\leq \frac{1}{\gamma^p}\left(\sup_{x>0} x^p e^{-x}\right)\left(\frac{\sqrt{2}C}{2^p}\right)^j.\end{aligned}$$

It follows that

$$\|u_j\|_{L^2(\tau_{j-1},\tau_j;L^2(\Omega))} \leq e^{\beta}\frac{1}{\sqrt{T}\gamma^p}\left(\sup_{x>0} x^p e^{-x}\right)\left(\frac{\sqrt{2}C}{2^p}\right)^j \left\|y^0\right\|_{L^2(\Omega)}. \tag{2.57}$$

Since $\bigcup_{j\in\mathbb{N}^*}[\tau_{j-1},\tau_j] = [0,T]$, we can define a function $u : (0,T)\times\Omega \longrightarrow \mathbb{R}$ by

$$u(t,\cdot) = u_j(t,\cdot), \quad \text{a.e. } t \in (\tau_{j-1},\tau_j).$$

Thanks to (2.57) we have

$$\sum_{j=1}^{+\infty}\|u_j\|^2_{L^2(\tau_{j-1},\tau_j;L^2(\Omega))} \leq e^{2\beta}\frac{1}{T\gamma^{2p}}\left(\sup_{x>0} x^p e^{-x}\right)^2\left(\sum_{j=1}^{+\infty}\left(\frac{\sqrt{2}C}{2^p}\right)^{2j}\right)\left\|y^0\right\|^2_{L^2(\Omega)}, \tag{2.58}$$

where the series converges thanks to the choice (2.56) of $p$. Thus, $u \in L^2(0,T;L^2(\Omega))$. Let then $y \in C^0([0,T];L^2(\Omega))$ be the corresponding solution to (2.1). By uniqueness of the solution to (2.1) on $(\tau_{j-1},\tau_j)$, we have

$$y(t) = y_j(t), \quad \forall t \in [\tau_{j-1},\tau_j].$$

In particular $y(\tau_j) = y_j(\tau_j)$. Combined with (2.54) we obtain that $y(\tau_j) \to 0$. But $y$ is continuous and $\tau_j \to T$ so that $y(\tau_j) \to y(T)$. By uniqueness of the limit it comes $y(T) = 0$. Finally, thanks to the choices of $\beta$ and $\gamma$ (see (2.55)), we see from (2.58) that we have the estimate (2.41) for $T$ small enough, say $T < 1$ (which is sufficient by Remark 2.5.2).

## 2.6 Bibliographical notes

**Method of moments.** A different and more elementary approach for the construction of a biorthogonal family to the exponentials is possible (see e.g. [Boy17, Section IV.1.2]). This other approach is slightly more general since it only assumes (2.30) while here we used the more precise asymptotic $\lambda_k \sim rk^2 + O(k)$ for some $r > 0$. However, this second approach does not give the estimate $Ce^{C/T}$ of the control cost and it does not provide the regularity $C^\infty([0,T])$ of the control either. For other material on the method of moments we also refer to the excellent textbook [You80].

**Lebeau-Robbiano method.** The presentation in Section 2.5 follows the remarkable clear presentation done in [Boy17, Section IV.2] from which we added extra computations to obtain the estimate $Ce^{C/T}$ of the cost of control. Note that we did not use the Weyl's law for the asymptotic of the eigenvalues of the Dirichlet Laplacian, which makes the proof more elementary. A proof of the Lebeau-Robbiano inequality (Theorem 2.5.3) can also be found in [Boy17, Section IV.2].

**Other PDEs.** For the controllability of other types of PDEs such as the transport equation, the wave equation, the Schrödinger equation and many more, we refer to [Cor07]. For a semigroup approach to the controllability of PDEs, we refer to [TW09].

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