Introduction to linear control theory

Lecture notes, Shandong University

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Contents

Chapter 1

Controllability of time-invariant linear O.D.E.s

1.1 Introduction

In this chapter we focus on the $n \times n$ time-invariant linear O.D.E.

$$\begin{cases} \dfrac{d}{dt}y &= Ay + Bu, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1.1}$$

where:

- $T > 0$ is a given time called time of control,
- $y^0 = (y_1^0, \ldots, y_n^0)$ is the initial data,
- $y = (y_1, \ldots, y_n)$ is the state,
- $A \in \mathbb{R}^{n \times n}$ is a matrix that couples the equations of the system,
- $u = (u_1, \ldots, u_m)$ are at our disposal, they are the so-called controls,
- $B \in \mathbb{R}^{n \times m}$ is a matrix that localizes the controls.

We recall that (1.1) is well-posed: for every $y^0 \in \mathbb{R}^n$ and every $u \in L^2(0,T)^m$, there exists a unique solution $y \in H^1(0,T)^n$ to the system (1.1) given by the Duhamel's formula

$$y(t) = e^{tA}y^0 + \int_0^t e^{(t-s)A}Bu(s)\,ds, \quad \forall t \in [0,T]. \tag{1.2}$$

Note in particular that

$$y \in C^0([0,T])^n,$$

which is crucial to define the different notions of controllability. Finally, note that

$$\|y(t)\| \leq C\left(\|y^0\| + \|u\|_{L^2(0,T)^m}\right), \quad \forall t \in [0,T], \tag{1.3}$$

for some $C > 0$ that does not depend on y^0 nor on u.

Definition 1.1.1 (Controllability)**.** We say that the system (1.1) is:

(i) exactly controllable in time T if, for every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$y(T) = y^1.$$

(ii) null-controllable in time T if the above property holds for $y^1 = 0$.

(iii) approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$\|y(T) - y^1\| \leq \varepsilon.$$

Example 1.1.2. *If $m = n$ and $B = \mathrm{Id}$, then (1.1) is exactly controllable in time T for every $T > 0$. Indeed, it is enough to take any smooth function y with $y(0) = y^0$ and $y(T) = y^1$ and set $u = \frac{d}{dt}y - Ay$.*

Remark 1.1.3. Clearly, exact controllability in time T implies null and approximate controllability in the same time T.

Remark 1.1.4. Let us consider the nonhomogeneous system

$$\begin{cases} \dfrac{d}{dt}y &= Ay + Bu + f(t), \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1.4}$$

where $f \in L^2(0,T)^n$. Then, we can define the corresponding notions of controllability exactly as in Definition 1.1.1, where instead y is now the solution to (1.4). It turns out that, if (1.1) is exactly controllable in time T, then (1.4) is exactly controllable in time T for every $f \in L^2(0,T)^n$ (the converse being obvious, we see that it is enough to only study the exact controllability of (1.1)). Indeed, firstly we consider the nonhomogeneous free system (that is without controls):

$$\begin{cases} \dfrac{d}{dt}\bar{y} &= A\bar{y} + f(t), \quad t \in (0,T), \\ \bar{y}(0) &= y^0, \end{cases}$$

and then we take a control that steers in time T the solution to (1.1) from 0 to $y^1 - \bar{y}(T)$.

Let us now reformulate the different notions of controllability. To this goal we introduce the linear operators

$$\begin{array}{rcrcl} F_T & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & y^0 & \longmapsto & \overline{y}(T), \end{array}$$

where $\overline{y}$ is the solution to the free system:

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\overline{y} & = & A\overline{y}, \quad t \in (0,T), \\ \overline{y}(0) & = & y^0, \end{array} \right.$$

and

$$\begin{array}{rcccl} G_T & : & L^2(0,T)^m & \longrightarrow & \mathbb{R}^n \\ & & u & \longmapsto & \widehat{y}(T), \end{array} \tag{1.5}$$

where $\hat{y}$ is the solution to the nonhomogeneous system with zero initial data:

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\widehat{y} & = & A\widehat{y} + Bu, \quad t \in (0,T), \\ \widehat{y}(0) & = & 0. \end{array} \right.$$

With these notations, we have

$$\begin{aligned} y(T) &= \overline{y}(T) + \hat{y}(T) \\ &= F_T y^0 + G_T u, \end{aligned} \tag{1.6}$$

where y is the solution to (1.1). It follows that:

(i) (1.1) is exactly controllable in time T if, and only if,

$$\operatorname{Im} G_T = \mathbb{R}^n. \tag{1.7}$$

(ii) (1.1) is null-controllable in time T if, and only if,

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T. \tag{1.8}$$

(iii) (1.1) is approximately controllable in time T if, and only if,

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n, \tag{1.9}$$

where $\overline{\operatorname{Im} G_T}$ denotes the closure of the set $\operatorname{Im} G_T$.

As a consequence of these reformulations we see that all the notions of controllability are equivalent for the finite dimensional system (1.1):

Proposition 1.1.5. *Let $T > 0$. The following statements are equivalent:*

(i) (1.1) *is exactly controllable in time* T.

(ii) (1.1) *is null-controllable in time* T.

(iii) (1.1) *is approximately controllable in time* T.

Therefore, from now on, we shall only say "controllable in time T*".*

Proof. Since $\operatorname{Im} F_T = \mathbb{R}^n$, it is clear that (1.7) and (1.8) are equivalent. On the other hand, (1.7) and (1.9) are clearly equivalent since $\operatorname{Im} G_T$ is a finite dimensional subspace and therefore it is closed. □

Remark 1.1.6. We arbitrarily chose to consider controls which are in $L^2(0,T)^m$ but let us mention that we can actually consider any dense subspace of $L^2(0,T)^m$ as control set. Indeed, for any subspace $V \subset L^2(0,T)^m$, we have

$$\operatorname{Im} G_{T|\overline{V}} \subset \overline{\operatorname{Im} G_{T|V}} = \operatorname{Im} G_{T|V},$$

where the inclusion holds because G_T is continuous (see (1.3)) and the equality holds because $\operatorname{Im} G_{T|V}$ is finite dimensional. In particular, if there exists a control which is barely in $L^2(0,T)^m$, then there exists as well a control which is smooth, say in $C_c^\infty(0,T)^m$.

1.2 Duality

Since $G_T \in \mathcal{L}(L^2(0,T)^m, \mathbb{R}^n)$ thanks to (1.3), we have

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n \quad \Longleftrightarrow \quad \ker G_T^* = \{0\}. \tag{1.10}$$

Thus, we want compute G_T^*. To this end we introduce the so-called adjoint system of (1.1), that is

$$\left\{ \begin{array}{rcl} -\dfrac{d}{dt} z & = & A^* z, \quad t \in (0,T), \\ z(T) & = & z^1, \end{array} \right. \tag{1.11}$$

where $z^1 \in \mathbb{R}^n$. Then, multiplying (1.1) by z and integrating by parts we obtain the following fundamental relation:

$$y(T) \cdot z^1 - y^0 \cdot z(0) = \int_0^T u(t) \cdot B^* z(t)\, dt, \tag{1.12}$$

valid for every $y^0 \in \mathbb{R}^n$, $z^1 \in \mathbb{R}^n$ and $u \in L^2(0,T)^m$. In (1.12) and in the sequel, $\cdot$ denotes the inner product (in $\mathbb{R}^n$ or in $\mathbb{R}^m$). Thanks to (1.12) we readily see that

$$\begin{array}{rcccc} G_T^* & : & \mathbb{R}^n & \longrightarrow & L^2(0,T)^m \\ & & z^1 & \longmapsto & B^* z. \end{array} \tag{1.13}$$

Using (1.10), we have obtained the following fundamental result:

Theorem 1.2.1 (Duality). (1.1) *is controllable in time T if, and only if,*

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* z(t) = 0, \quad \forall t \in [0, T]\Big) \Longrightarrow z^1 = 0. \tag{1.14}$$

Remark 1.2.2. Clearly, (1.14) is equivalent to

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* \tilde{z}(t) = 0, \quad \forall t \in [0, T]\Big) \Longrightarrow z^1 = 0,$$

where $\tilde{z}(t) = z(T - t)$. But $\tilde{z}$ is analytic on $(0, +\infty)$. Thus, (1.14) holds if, and only if,

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* \tilde{z}(t) = 0, \quad \forall t \in [0, +\infty)\Big) \Longrightarrow z^1 = 0.$$

Therefore, the controllability of (1.1) does not depend on the time of control T. In other words, if there exists $T > 0$ such that (1.1) is controllable in time T, then, for every $T > 0$, (1.1) is controllable in time T. For this reason, in the sequel we shall only say that (1.1) is "controllable".

Remark 1.2.3. The strength of the duality is that it reduces the task of proving an existence result (existence of a control) to the task of proving a uniqueness result, which is often easier to handle.

1.3 Conditions of controllability

1.3.1 Gramian of controllability

Theorem 1.3.1. *Let $T > 0$. (1.1) is controllable if, and only if, the $n \times n$ matrix*

$$\Lambda_T = \int_0^T e^{(T-t)A} B B^* e^{(T-t)A^*} \, dt, \tag{1.15}$$

is invertible. Λ_T is called the Gramian of controllability or HUM operator.

Remark 1.3.2. Note that Λ_T is always symmetric and positive semi-definite. In particular, it is invertible if, and only if, it is positive definite. Now observe that Λ_T is positive definite if, and only if, there exists $C_T > 0$ such that

$$\|z^1\|^2 \leq C_T^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n. \tag{1.16}$$

This inequality is called observability inequality and the best constant $C_T > 0$ in (1.16) is called the control cost. We shall come back to this notion later on in Section **??**.

Proof. By Theorem 1.2.1, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Clearly, this is equivalent to

$$\ker G_T G_T^* = \{0\}.$$

By definition of G_T (see (1.5)) and computation of G_T^* (see (1.13)) we readily see that $G_T G_T^* = \Lambda_T$. □

1.3.2 Kalman rank condition

Lemma 1.3.3. *For every $T > 0$, we have*

$$\ker G_T^* = \left(\mathrm{Im}\,\left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

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