Introduction to linear control theory

Lecture notes, Shandong University

Guillaume Olive[1]

March 17, 2017

[1]E-mail: math.golive@gmail.com

Contents

Chapter 1

Controllability of time-invariant linear O.D.E.s

1.1 Introduction

In this chapter we focus on the $n \times n$ time-invariant linear O.D.E.

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}y & = & Ay + Bu, \quad t \in (0,T), \\ y(0) & = & y^0, \end{array} \right. \tag{1.1}$$

where:

- $T > 0$ is a given time called time of control,
- $y^0 = (y_1^0, \dots, y_n^0)$ is the initial data,
- $y = (y_1, \dots, y_n)$ is the state,
- $A \in \mathbb{R}^{n \times n}$ is a matrix that couples the equations of the system,
- $u = (u_1, \dots, u_m)$ are at our disposal, they are the so-called controls,
- $B \in \mathbb{R}^{n \times m}$ is a matrix that localizes the controls.

We recall that (1.1) is well-posed: for every $y^0 \in \mathbb{R}^n$ and every $u \in L^2(0,T)^m$, there exists a unique solution $y \in H^1(0,T)^n$ to the system (1.1) given by the Duhamel's formula

$$y(t) = e^{tA}y^0 + \int_0^t e^{(t-s)A}Bu(s)\,ds, \quad \forall t \in [0,T]. \tag{1.2}$$

Note in particular that

$$y \in C^0([0,T])^n,$$

which is crucial to define the different notions of controllability. Finally, note that

$$\|y(t)\| \leq C\left(\|y^0\| + \|u\|_{L^2(0,T)^m}\right), \quad \forall t \in [0,T], \tag{1.3}$$

for some $C > 0$ that does not depend on y^0 nor on u.

Definition 1.1.1 (Controllability)**.** We say that the system (1.1) is:

(i) exactly controllable in time T if, for every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$y(T) = y^1.$$

(ii) null-controllable in time T if the above property holds for $y^1 = 0$.

(iii) approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$\|y(T) - y^1\| \leq \varepsilon.$$

Example 1.1.2. *If $m = n$ and $B = \mathrm{Id}$, then (1.1) is exactly controllable in time T for every $T > 0$. Indeed, it is enough to take any smooth function y with $y(0) = y^0$ and $y(T) = y^1$ and set $u = \frac{d}{dt}y - Ay$.*

Remark 1.1.3. Clearly, exact controllability in time T implies null and approximate controllability in the same time T.

Remark 1.1.4. Let us consider the nonhomogeneous system

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}y & = & Ay + Bu + f(t), \quad t \in (0,T), \\ y(0) & = & y^0, \end{array}\right. \tag{1.4}$$

where $f \in L^2(0,T)^n$. Then, we can define the corresponding notions of controllability exactly as in Definition 1.1.1, where instead y is now the solution to (1.4). It turns out that, if (1.1) is exactly controllable in time T, then (1.4) is exactly controllable in time T for every $f \in L^2(0,T)^n$ (the converse being obvious, we see that it is enough to only study the exact controllability of (1.1)). Indeed, firstly we consider the nonhomogeneous free system (that is without controls):

$$\left\{\begin{array}{rcl} \dfrac{d}{dt}\bar{y} & = & A\bar{y} + f(t), \quad t \in (0,T), \\ \bar{y}(0) & = & y^0, \end{array}\right.$$

and then we take a control that steers in time T the solution to (1.1) from 0 to $y^1 - \bar{y}(T)$.

Let us now reformulate the different notions of controllability. To this goal we introduce the linear operators

$$\begin{array}{cccc} F_T & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & y^0 & \longmapsto & \overline{y}(T), \end{array}$$

where $\overline{y}$ is the solution to the free system:

$$\begin{cases} \dfrac{d}{dt}\overline{y} & = & A\overline{y}, \quad t \in (0,T), \\ \overline{y}(0) & = & y^0, \end{cases}$$

and

$$\begin{array}{cccc} G_T & : & L^2(0,T)^m & \longrightarrow & \mathbb{R}^n \\ & & u & \longmapsto & \widehat{y}(T), \end{array} \tag{1.5}$$

where $\hat{y}$ is the solution to the nonhomogeneous system with zero initial data:

$$\begin{cases} \dfrac{d}{dt}\widehat{y} & = & A\widehat{y} + Bu, \quad t \in (0,T), \\ \widehat{y}(0) & = & 0. \end{cases}$$

With these notations, we have

$$\begin{aligned} y(T) & = \overline{y}(T) + \hat{y}(T) \\ & = F_T y^0 + G_T u, \end{aligned} \tag{1.6}$$

where y is the solution to (1.1). It follows that:

(i) (1.1) is exactly controllable in time T if, and only if,

$$\operatorname{Im} G_T = \mathbb{R}^n. \tag{1.7}$$

(ii) (1.1) is null-controllable in time T if, and only if,

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T. \tag{1.8}$$

(iii) (1.1) is approximately controllable in time T if, and only if,

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n, \tag{1.9}$$

where $\overline{\operatorname{Im} G_T}$ denotes the closure of the set $\operatorname{Im} G_T$.

As a consequence of these reformulations we see that all the notions of controllability are equivalent for the finite dimensional system (1.1):

Proposition 1.1.5. *Let $T > 0$. The following statements are equivalent:*

(i) (1.1) *is exactly controllable in time* T.

(ii) (1.1) *is null-controllable in time* T.

(iii) (1.1) *is approximately controllable in time* T.

Therefore, from now on, we shall only say "controllable in time T*".*

Proof. Since $\operatorname{Im} F_T = \mathbb{R}^n$, it is clear that (1.7) and (1.8) are equivalent. On the other hand, (1.7) and (1.9) are clearly equivalent since $\operatorname{Im} G_T$ is a finite dimensional subspace and therefore it is closed. □

Remark 1.1.6. We arbitrarily chose to consider controls which are in $L^2(0,T)^m$ but let us mention that we can actually consider any dense subspace of $L^2(0,T)^m$ as control set. Indeed, for any subspace $V \subset L^2(0,T)^m$, we have

$$\operatorname{Im} G_{T|\overline{V}} \subset \overline{\operatorname{Im} G_{T|V}} = \operatorname{Im} G_{T|V},$$

where the inclusion holds because G_T is continuous (see (1.3)) and the equality holds because $\operatorname{Im} G_{T|V}$ is finite dimensional. In particular, if there exists a control which is barely in $L^2(0,T)^m$, then there exists as well a control which is smooth, say in $C_c^\infty(0,T)^m$.

1.2 Duality

Since $G_T \in \mathcal{L}(L^2(0,T)^m, \mathbb{R}^n)$ thanks to (1.3), we have

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n \quad \Longleftrightarrow \quad \ker G_T^* = \{0\}\,. \tag{1.10}$$

Thus, we want compute G_T^*. To this end we introduce the so-called adjoint system of (1.1), that is

$$\begin{cases} -\dfrac{d}{dt} z & = & A^* z, \quad t \in (0,T), \\ z(T) & = & z^1, \end{cases} \tag{1.11}$$

where $z^1 \in \mathbb{R}^n$. Then, multiplying (1.1) by z and integrating by parts we obtain the following fundamental relation:

$$y(T) \cdot z^1 - y^0 \cdot z(0) = \int_0^T u(t) \cdot B^* z(t)\, dt, \tag{1.12}$$

valid for every $y^0 \in \mathbb{R}^n$, $z^1 \in \mathbb{R}^n$ and $u \in L^2(0,T)^m$. In (1.12) and in the sequel, $\cdot$ denotes the inner product (in $\mathbb{R}^n$ or in $\mathbb{R}^m$). Thanks to (1.12) we readily see that

$$\begin{array}{rccc} G_T^* & : & \mathbb{R}^n & \longrightarrow & L^2(0,T)^m \\ & & z^1 & \longmapsto & B^* z. \end{array} \tag{1.13}$$

Using (1.10), we have obtained the following fundamental result:

Theorem 1.2.1 (Duality). (1.1) *is controllable in time T if, and only if,*

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* z(t) = 0, \quad \forall t \in [0,T]\Big) \Longrightarrow z^1 = 0. \tag{1.14}$$

Remark 1.2.2. Clearly, (1.14) is equivalent to

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* \tilde{z}(t) = 0, \quad \forall t \in [0,T]\Big) \Longrightarrow z^1 = 0,$$

where $\tilde{z}(t) = z(T-t)$. But $\tilde{z}$ is analytic on $(0,+\infty)$. Thus, (1.14) holds if, and only if,

$$\forall z^1 \in \mathbb{R}^n, \quad \Big(B^* \tilde{z}(t) = 0, \quad \forall t \in [0,+\infty)\Big) \Longrightarrow z^1 = 0.$$

Therefore, the controllability of (1.1) does not depend on the time of control T. In other words, if there exists $T>0$ such that (1.1) is controllable in time T, then, for every $T>0$, (1.1) is controllable in time T. For this reason, in the sequel we shall only say that (1.1) is "controllable".

Remark 1.2.3. The strength of the duality is that it reduces the task of proving an existence result (existence of a control) to the task of proving a uniqueness result, which is often easier to handle.

1.3 Conditions of controllability

1.3.1 Gramian of controllability

Theorem 1.3.1. *Let $T>0$. (1.1) is controllable if, and only if, the $n \times n$ matrix*

$$\Lambda_T = \int_0^T e^{(T-t)A} B B^* e^{(T-t)A^*} \, dt, \tag{1.15}$$

is invertible. Λ_T is called the Gramian of controllability or HUM operator.

Remark 1.3.2. Note that Λ_T is always symmetric and positive semi-definite. In particular, it is invertible if, and only if, it is positive definite. Now observe that Λ_T is positive definite if, and only if, there exists $C_T > 0$ such that

$$\|z^1\|^2 \leq C_T^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n. \tag{1.16}$$

This inequality is called observability inequality and the best constant $C_T > 0$ in (1.16) is called the control cost. We shall come back to this notion later on in Section **??**.

Proof. By Theorem 1.2.1, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Clearly, this is equivalent to

$$\ker G_T G_T^* = \{0\}.$$

By definition of G_T (see (1.5)) and computation of G_T^* (see (1.13)) we readily see that $G_T G_T^* = \Lambda_T$. □

1.3.2 Kalman rank condition

Lemma 1.3.3. *For every $T > 0$, we have*

$$\ker G_T^* = \left(\operatorname{Im}\left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

Proof. By (1.13), $z^1 \in \ker G_T^*$ if, and only if,

$$B^* z(t) = 0, \quad \forall t \in [0,T], \tag{1.17}$$

where $z(t) = e^{(T-t)A^*} z^1$ is the solution to the adjoint system (1.11). Since z is analytic on $(0,T)$, we have (1.17) if, and only if, for some $0 < t_0 < T$,

$$\frac{d^k}{dt^k}(B^* z)(t_0) = 0, \quad \forall k \in \{0,1,\ldots\}.$$

Computing $B^* z$ that gives

$$B^*(A^*)^k z^1 = 0, \quad \forall k \in \{0,1,\ldots\}.$$

By the Cayley-Hamilton theorem, this is equivalent to

$$B^*(A^*)^k z^1 = 0, \quad \forall k \in \{0,1,\ldots,n-1\}.$$

To summarize, $z^1 \in \ker G_T^*$ if, and only, if

$$z^1 \in \ker \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix}.$$

To conclude, observe that

$$\ker \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix} = \ker\left(B|AB|\cdots|A^{n-1}B\right)^* = \left(\operatorname{Im}\left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

□

An immediate consequence of Theorem 1.2.1 and Lemma 1.3.3 is the following fundamental result:

Theorem 1.3.4 (Kalman rank condition)**.** (1.1) *is controllable if, and only if,*

$$\text{rank}\,(B|AB|\cdots|A^{n-1}B) = n. \tag{1.18}$$

Observe that, as expected (see Remark 1.2.2), the condition (1.18) does not depend on the time of control T.

The Kalman rank condition is an easy checkable condition for the controllability as it is shown on the following example.

Example 1.3.5. *The* 2×2 *system*

$$\begin{cases} \dfrac{d}{dt}y_1 = a_{11}y_1 + a_{12}y_2 + u, \\ \dfrac{d}{dt}y_2 = a_{21}y_1 + a_{22}y_2, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, \end{cases} \qquad t \in (0,T),$$

is controllable if, and only if,

$$a_{21} \neq 0.$$

Remark 1.3.6. Thanks to the Kalman rank condition we also see that we can fix the end-points of the control (and of its derivatives). Indeed, say that we look for controls u such that, in addition,

$$u(0) = u^0, \quad u(T) = u^1,$$

for some $u^0, u^1 \in \mathbb{R}^m$. Then, to this end we consider u as a new variable and we introduce the $(n+m)\times(n+m)$ augmented system

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, \\ \frac{d}{dt}u = v, \\ y(0) = y^0, \quad u(0) = u^0, \end{cases} \qquad t \in (0,T),$$

where v is now the control. We easily check that this system satisfies the associated Kalman rank condition.

Actually, we even have a stronger result than Theorem 1.3.4 since we can give a precise characterization of the reachable states:

Theorem 1.3.7. *Let* $y^0, y^1 \in \mathbb{R}^n$ *and* $T > 0$ *be fixed. There exists* $u \in L^2(0,T)^m$ *such that the corresponding solution* y *to* (1.1) *satisfies* $y(T) = y^1$ *if, and only if,*

$$y^1 - e^{TA}y^0 \in \text{Im}\,\left(B|AB|\cdots|A^{n-1}B\right).$$

Proof. Using (1.6) we readily see that there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to (1.1) satisfies $y(T) = y^1$ if, and only if,

$$y^1 - e^{TA}y^0 \in \operatorname{Im} G_T.$$

Since $\operatorname{Im} G_T = (\ker G_T^*)^\perp$, the result follows from Lemma 1.3.3. □

There is a canonical form of controllable systems.

Proposition 1.3.8 (Canonical form of Brunovski)**.** *Assume that $m = 1$. Assume that* (1.18) *holds and let $K = (B|AB|\cdots|A^{n-1}B)$ (note that $K \in \mathbb{R}^{n\times n}$). Then,*

$$K^{-1}AK = \tilde{A}, \quad K^{-1}B = \tilde{B},$$

with

$$\tilde{A} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \alpha_0 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{n-1} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \tag{1.19}$$

where $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{R}$ are the coefficients of the characteristic polynomial of A, namely $p(\lambda) = \lambda^n - \alpha_{n-1}\lambda^{n-1} - \ldots - \alpha_0$, $\lambda \in \mathbb{C}$.

Proof. The proof is a simple computation of $K\tilde{A}$ and $K\tilde{B}$. □

It is worth mentioning that, once we know the "good" condition for the controllability (namely, (1.18)), there exists a direct proof of Theorem 1.3.4. By direct proof we mean a proof that is not using the duality at all. It is based on Proposition 1.3.8 and the following result, that we shall prove in a self-contained way:

Proposition 1.3.9. *Let $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{R}$. The system*

$$\begin{cases} \dfrac{d}{dt}y &= \tilde{A}y + \tilde{B}u, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1.20}$$

where $\tilde{A}$ and $\tilde{B}$ are given by (1.19)*, is controllable.*

Remark 1.3.10. Combining Proposition 1.3.8 with 1.3.9 this gives a direct proof of the implication "$\Longleftarrow$" in Theorem 1.3.4 for $m = 1$.

Proof of Proposition 1.3.9 (without using Theorem 1.3.4). We give a direct proof. We recall that it is sufficient to only consider the target $y^1 = 0$ (see Proposition 1.1.5). Let $\overline{y}$ be the free solution to (1.20), that is the solution to (1.20) with $u = 0$. Let us introduce a cut-off function $\eta \in C^\infty([0,T])$ such that

$$\eta = 1 \text{ on } [0, T/3], \quad \eta = 0 \text{ on } [2T/3, T].$$

Observe that, because of the structure (1.19), the last equation of (1.20) is

$$\frac{d}{dt} y_n = y_{n-1} + \alpha_{n-1} y_n.$$

We set

$$y_n = \eta \overline{y}_n.$$

Then, we have no choice for y_{n-1} but to set

$$y_{n-1} = \frac{d}{dt} y_n - \alpha_{n-1} y_n.$$

By induction, we have to set

$$y_k = \frac{d}{dt} y_{k+1} - \alpha_k y_n, \quad \forall k \in \{n-2, \dots, 1\},$$

and then

$$u = \frac{d}{dt} y_1 - \alpha_0 y_n.$$

Finally, thanks to the definition of η, note that

$$\forall k \in \{1, \dots, n\}, \quad \begin{cases} y_k = \overline{y}_k & \text{on } [0, T/3], \\ y_k = 0 & \text{on } [2T/3, T], \end{cases}$$

so that

$$y(0) = y^0, \quad y(T) = 0.$$

□

1.3.3 Fattorini-Hautus test

There is another important characterization of the controllability, which is a dual version of the Kalman rank condition:

Theorem 1.3.11 (Fattorini-Hautus test)**.** (1.1) *is controllable if, and only if,*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{1.21}$$

Remark 1.3.12. Theorem 1.3.11 shows in particular that the following condition is necessary for the controllability:

$$\dim\ker(\lambda - A^*) \leq m, \quad \forall \lambda \in \mathbb{C}.$$

Indeed, assume that there exists a linearly independent family $\phi_1, \ldots, \phi_{m+1}$ of $\ker(\lambda - A^*)$. Then, $B^*\phi_1, \ldots, B^*\phi_{m+1}$ is linearly dependent as $B^* \in \mathbb{R}^{m\times n}$. Thus, there exists $(\alpha_1, \ldots, \alpha_{m+1}) \neq (0, \ldots, 0)$ such that $\sum_{k=1}^{m+1} \alpha_k B^*\phi_k = 0$. Let $z^1 = \sum_{k=1}^{m+1} \alpha_k \phi_k$. Then, $B^*z^1 = 0$. But $z^1 \in \ker(\lambda - A^*)$. Therefore, (1.21) implies that $z^1 = 0$, that is $\alpha_1 = \ldots = \alpha_{m+1} = 0$, a contradiction.

Proof. By Theorem 1.2.1, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Assume that $\ker G_T^* = \{0\}$. Let $z^1 \in \ker(\lambda - A^*) \cap \ker B^*$. Then, $z(t) = e^{\lambda(T-t)}z^1$ and $B^*z(t) = e^{\lambda(T-t)}B^*z^1 = 0$ for every $t \in [0,T]$. Therefore, $z^1 = 0$ by assumption. Conversely, assume that $\ker G_T^* \neq \{0\}$. Let us first prove that:

(i) $\ker G_T^* \subset \ker B^*$.

(ii) $A^*(\ker G_T^*) \subset \ker G_T^*$.

Let $z^1 \in \ker G_T^*$. Then,

$$B^*z(t) = 0, \quad \forall t \in [0,T].$$

Taking $t = T$ we obtain $B^*z^1 = 0$, that is $z^1 \in \ker B^*$. On the other hand, taking the derivative we obtain

$$B^*e^{(T-t)A^*}A^*z^1 = 0, \quad \forall t \in [0,T],$$

that is $A^*z^1 \in \ker G_T^*$. Consequently, by (ii) we see the restriction of A^* to $\ker G_T^*$ is a linear operator from the finite dimensional space $\ker G_T^*$ into itself and, since $\ker G_T^* \neq \{0\}$, therefore possesses at least one complex eigenvalue. Since in addition by (i) we have $\ker G_T^* \subset \ker B^*$, this shows that there exist $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{R}^n$ with $\phi \neq 0$ such that

$$A^*\phi = \lambda\phi, \quad B^*\phi = 0.$$

This proves that (1.21) fails. □

1.3.4 Partial controllability

Sometimes we want to control not all but only some components of the system (1.1). This leads to the notion of partial controllability (also called output controllability in the literature).

Definition 1.3.13 (Partial controllability)**.** Let $P \in \mathbb{R}^{p\times n}$ with $p \in \mathbb{N}^*$. We say that the system (1.1) is partially controllable if, for every $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^p$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$Py(T) = y^1.$$

One can take for instance the projection on the first p components:

$$\begin{array}{cccc} P & : & \mathbb{R}^p \times \mathbb{R}^{n-p} & \longrightarrow & \mathbb{R}^p \\ & & (y_1, y_2) & \longmapsto & y_1, \end{array}$$

where $p \in \{1, \ldots, n-1\}$ is the number of components we would like to control.

Mimicking the procedure developed in the previous sections, we see that (1.1) is partially controllable if, and only if,

$$\overline{\operatorname{Im} PG_T} = \mathbb{R}^n.$$

This is equivalent to

$$\ker G_T^* P^* = \{0\}.$$

Thanks to the expression of G_T^* (see (1.13)) we see that this is also equivalent to

$$\forall z^1 \in \mathbb{R}^p, \quad \Big(B^* z(t) = 0, \quad \text{a.e. } t \in (0,T)\Big) \Longrightarrow z^1 = 0,$$

where z is the solution to the following adjoint system:

$$\left\{\begin{array}{rcl} -\dfrac{d}{dt}z & = & A^* z, \quad t \in (0,T), \\ z(T) & = & P^* z^1. \end{array}\right.$$

Reproducing the proof of Lemma 1.3.3 we easily obtain the following result:

Theorem 1.3.14 (Kalman rank condition)**.** (1.1) *is partially controllable if, and only if,*

$$\operatorname{rank}(PB|PAB|\cdots|PA^{n-1}B) = p.$$

1.3.5 Higher order O.D.E.s

An interesting consequence of Theorem 1.3.4 is that it also gives a characterization of the controllability of higher order systems.

Let $y^0, \dot{y}^0 \in \mathbb{R}^n$ and let us consider the second order system:

$$\left\{\begin{array}{l} \dfrac{d^2}{dt^2}y = Ay + Bu, \quad t \in (0,T), \\ y(0) = y^0, \quad \dfrac{d}{dt}y(0) = \dot{y}^0. \end{array}\right. \tag{1.22}$$

Firstly, we should point out that there are a priori several ways to define to controllability of (1.22). Do we want to achieve $y(T) = \frac{d}{dt}y(T) = 0$ or only $y(T) = 0$ for instance ? Note that the first goal is more physical since $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ is a stable state of the system (1.22) (while $(0, \dot{y}^1)$ is not) and, therefore, once the system has reached this state, it stays at this state without any additional control required. However, we will study the two situations as both are of mathematical interest.

Definition 1.3.15 (Controllability)**.** We say that the system (1.22) is:

(i) controllable if, for every $y^0, \dot{y}^0 \in \mathbb{R}^n$ and $y^1, \dot{y}^1 \in \mathbb{R}^n$, there exists $u \in L^2(0, T)^m$ such that the corresponding solution y to system (1.22) satisfies

$$y(T) = y^1, \quad \frac{d}{dt}y(T) = \dot{y}^1.$$

(ii) partially controllable if, for every $y^0, \dot{y}^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0, T)^m$ such that the corresponding solution y to system (1.22) satisfies

$$y(T) = y^1.$$

Of course the notion of partial controllability for (1.22) coincides with the notion of partial controllability of Section 1.3.4 for an underlying first order system. Surprisingly enough, it turns out that the notions of controllability and partial controllability for (1.22) are equivalent.

Theorem 1.3.16. *The following statements are equivalent:*

(i) (1.22) *is controllable.*

(ii) (1.22) *is partially controllable.*

(iii) $\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n$.

Proof. "(i) $\Longleftrightarrow$ (iii)". Introducing the new variable

$$\tilde{y} = \begin{pmatrix} y \\ \frac{d}{dt}y \end{pmatrix} \in \mathbb{R}^{2n},$$

and

$$\tilde{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ A & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n}, \quad \tilde{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathbb{R}^{2n \times m},$$

we see that the $n \times n$ second order system (1.22) is controllable if, and only if, so is the following $2n \times 2n$ first order system:

$$\begin{cases} \frac{d}{dt}\tilde{y} &= \tilde{A}\tilde{y} + \tilde{B}u, \quad t \in (0, T), \\ \tilde{y}(0) &= \tilde{y}^0. \end{cases} \tag{1.23}$$

By Theorem 1.3.4, the controllability of (1.23) is equivalent to the corresponding Kalman rank condition, that is

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2n.$$

A computation shows that

$$(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = \begin{pmatrix} 0 & B & 0 & AB & \cdots & 0 & A^{n-1}B \\ B & 0 & AB & 0 & \cdots & A^{n-1}B & 0 \end{pmatrix}. \tag{1.24}$$

Therefore,

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2\operatorname{rank}(B|AB|\cdots|A^{n-1}B).$$

"(ii) $\Longleftrightarrow$ (iii)". Let us introduce

$$\begin{array}{rcccl} P & : & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & (y_1, y_2) & \longmapsto & y_1. \end{array}$$

Then, the $n\times n$ second order system (1.22) is partially controllable if, and only if, the $2n\times 2n$ first order system (1.23) is partially controllable. By Theorem 1.3.14, this is equivalent to the corresponding Kalman rank condition, namely

$$\operatorname{rank}(P\tilde{B}|P\tilde{A}\tilde{B}|\cdots|P\tilde{A}^{2n-1}\tilde{B}) = n.$$

Thanks to (1.24) we readily see that it is equivalent to $\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n$. □

1.4 Optimal controls

1.4.1 Control of minimal L^2-norm

Assume that (1.1) is controllable. A priori there is no reason for a control to be unique. Let $y^0, y^1 \in \mathbb{R}^n$ and let us introduce the corresponding set of admissible controls

$$U = \left\{ u \in L^2(0,T)^m, \quad y(T) = y^1 \right\}.$$

We consider the minimization problem

$$\min_{u\in U} \frac{1}{2}\|u\|^2_{L^2(0,T)^m}.$$

A solution of this problem will be called a control of minimal L^2-norm.

Theorem 1.4.1 (L^2-optimal control)**.** *Assume that* (1.1) *is controllable. Then, for every $y^0, y^1 \in \mathbb{R}^n$, there exists a unique control of minimal L^2- norm and it is given by*

$$u_{\text{opt}}(t) = B^* e^{(T-t)A^*} \Lambda_T^{-1} \left(y^1 - e^{TA} y^0 \right), \tag{1.25}$$

where $\Lambda_T \in \mathbb{R}^{n\times n}$ is the Gramian of controllability (see (1.15)*). The control u_{opt} is also called the HUM control.*

Remark 1.4.2. Note that the control u_{opt} is analytic on $\mathbb{R}$.

Lemma 1.4.3 (Hilbert projection theorem)**.** *Let H be a Hilbert space. Let $C \subset H$ be a nonempty closed convex. For every $x \in H$, there exists a unique $p \in C$ such that*

$$\|x-p\| = \min_{y \in C} \|x-y\|.$$

Moreover, p is the unique element of C that satisfies

$$\langle x-p, y-p\rangle \leq 0, \quad \forall y \in C.$$

Proof of Theorem 1.4.1. Firstly, observe that U is not empty by assumption. Let then $u_0 \in U$. We easily see that

$$U = u_0 + \ker G_T.$$

Therefore, U is an affine subspace of $\mathbb{R}^n$. In particular, it is a closed convex and, by Lemma 1.4.3, there exists a unique $u_{\text{opt}} \in U$ (the projection of 0 on U) such that

$$\|u_{\text{opt}}\| = \min_{u \in U} \|u\|.$$

Moreover, we have

$$\langle u_{\text{opt}}, u_{\text{opt}} - u\rangle_{L^2} \leq 0, \quad \forall u \in U.$$

Since $U = u_{\text{opt}} + \ker G_T$, this gives

$$\langle u_{\text{opt}}, v\rangle_{L^2} = 0, \quad \forall v \in \ker G_T.$$

Thus,

$$u_{\text{opt}} \in (\ker G_T)^{\perp} = \overline{\operatorname{Im} G_T^*}.$$

Then, there exists $(z_n^1)_n$ such that

$$G_T^* z_n^1 \xrightarrow[n\to+\infty]{} u_{\text{opt}}. \tag{1.26}$$

But $(z_n^1)_n$ converges. Indeed, since $u_{\text{opt}} \in U$, we have

$$G_T u_{\text{opt}} = y^1 - F_T y^0.$$

Since G_T is a bounded operator, combined with (1.26) this gives

$$G_T G_T^* z_n^1 \xrightarrow[n\to+\infty]{} y^1 - F_T y^0.$$

By Theorem 1.15, we know that $G_T G_T^* = \Lambda_T$ is invertible. Thus,

$$z_n^1 \xrightarrow[n\to+\infty]{} \Lambda_T^{-1}\left(y^1 - F_T y^0\right).$$

Coming back to (1.26), we obtain that

$$u_{\text{opt}} = G_T^* \Lambda_T^{-1}\left(y^1 - F_T y^0\right).$$

The expressions of G_T^* (see (1.13)) and F_T finally give (1.25). □

References

[Boy17] Franck Boyer, *Controllability of parabolic pdes: old and new*, 2017, https://www.math.univ-toulouse.fr/~fboyer/_media/enseignements/m2_lecture_notes_fboyer.pdf.

[Cor07] J.-M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007. MR 2302744 (2008d:93001)

[LM67] E. B. Lee and L. Markus, *Foundations of optimal control theory*, John Wiley & Sons, Inc., New York-London-Sydney, 1967. MR 0220537

[Sei88] Thomas I. Seidman, *How violent are fast controls?*, Math. Control Signals Systems **1** (1988), no. 1, 89–95. MR 923278

[Son98] Eduardo D. Sontag, *Mathematical control theory*, second ed., Texts in Applied Mathematics, vol. 6, Springer-Verlag, New York, 1998, Deterministic finite-dimensional systems. MR 1640001

[TW09] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009. MR 2502023

[Zab08] Jerzy Zabczyk, *Mathematical control theory*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, An introduction, Reprint of the 1995 edition. MR 2348543 (2008e:49001)