Introduction to linear control theory

Lecture notes, Shandong University

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Contents

Chapter 1

Controllability of time-invariant linear O.D.E.s

1.1 Introduction

In this chapter we focus on the $n \times n$ time-invariant linear O.D.E.

$$\begin{cases} \dfrac{d}{dt}y &= Ay + Bu, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1.1}$$

where:

- $T > 0$ is a given time called time of control,
- $y^0 = (y_1^0, \ldots, y_n^0)$ is the initial data,
- $y = (y_1, \ldots, y_n)$ is the state,
- $A \in \mathbb{R}^{n\times n}$ is a matrix that couples the equations of the system,
- $u = (u_1, \ldots, u_m)$ are at our disposal, they are the so-called controls,
- $B \in \mathbb{R}^{n\times m}$ is a matrix that localizes the controls.

We recall that (1.1) is well-posed: for every $y^0 \in \mathbb{R}^n$ and every $u \in L^2(0,T)^m$, there exists a unique solution $y \in H^1(0,T)^n$ to the system (1.1) given by the Duhamel's formula

$$y(t) = e^{tA}y^0 + \int_0^t e^{(t-s)A}Bu(s)\,ds, \quad \forall t \in [0,T]. \tag{1.2}$$

Note in particular that

$$y \in C^0([0,T])^n,$$

which is crucial to define the different notions of controllability. Finally, note that

$$\|y(t)\| \leq C\left(\left\|y^0\right\| + \|u\|_{L^2(0,T)^m}\right), \quad \forall t \in [0,T], \tag{1.3}$$

for some $C > 0$ that does not depend on y^0 nor on u.

Definition 1.1.1 (Controllability)**.** We say that the system (1.1) is:

(i) exactly controllable in time T if, for every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$y(T) = y^1.$$

(ii) null-controllable in time T if the above property holds for $y^1 = 0$.

(iii) approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$\left\|y(T) - y^1\right\| \leq \varepsilon.$$

Example 1.1.2. *If $m = n$ and $B = \mathrm{Id}$, then (1.1) is exactly controllable in time T for every $T > 0$. Indeed, it is enough to take any smooth function y with $y(0) = y^0$ and $y(T) = y^1$ and set $u = \frac{d}{dt}y - Ay$.*

Remark 1.1.3. Clearly, exact controllability in time T implies null and approximate controllability in the same time T.

Remark 1.1.4. Let us consider the nonhomogeneous system

$$\begin{cases} \dfrac{d}{dt}y = Ay + Bu + f(t), & t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{1.4}$$

where $f \in L^2(0,T)^n$. Then, we can define the corresponding notions of controllability exactly as in Definition 1.1.1, where instead y is now the solution to (1.4). It turns out that, if (1.1) is exactly controllable in time T, then (1.4) is exactly controllable in time T for every $f \in L^2(0,T)^n$ (the converse being obvious, we see that it is enough to only study the exact controllability of (1.1)). Indeed, firstly we consider the nonhomogeneous free system (that is without controls):

$$\begin{cases} \dfrac{d}{dt}\bar{y} = A\bar{y} + f(t), & t \in (0,T), \\ \bar{y}(0) = y^0, \end{cases}$$

and then we take a control that steers in time T the solution to (1.1) from 0 to $y^1 - \bar{y}(T)$.

Let us now reformulate the different notions of controllability. To this goal we introduce the linear operators

$$\begin{array}{rcccl} F_T & : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & y^0 & \longmapsto & \overline{y}(T), \end{array}$$

where $\overline{y}$ is the solution to the free system:

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\overline{y} & = & A\overline{y}, \quad t \in (0,T), \\ \overline{y}(0) & = & y^0, \end{array} \right.$$

and

$$\begin{array}{rcccl} G_T & : & L^2(0,T)^m & \longrightarrow & \mathbb{R}^n \\ & & u & \longmapsto & \widehat{y}(T), \end{array} \tag{1.5}$$

where $\hat{y}$ is the solution to the nonhomogeneous system with zero initial data:

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\widehat{y} & = & A\widehat{y} + Bu, \quad t \in (0,T), \\ \widehat{y}(0) & = & 0. \end{array} \right.$$

With these notations, we have

$$\begin{aligned} y(T) &= \overline{y}(T) + \hat{y}(T) \\ &= F_T y^0 + G_T u, \end{aligned} \tag{1.6}$$

where y is the solution to (1.1). It follows that:

(i) (1.1) is exactly controllable in time T if, and only if,

$$\operatorname{Im} G_T = \mathbb{R}^n. \tag{1.7}$$

(ii) (1.1) is null-controllable in time T if, and only if,

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T. \tag{1.8}$$

(iii) (1.1) is approximately controllable in time T if, and only if,

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n, \tag{1.9}$$

where $\overline{\operatorname{Im} G_T}$ denotes the closure of the set $\operatorname{Im} G_T$.

As a consequence of these reformulations we see that all the notions of controllability are equivalent for the finite dimensional system (1.1):

Proposition 1.1.5. *Let $T > 0$. The following statements are equivalent:*

(i) (1.1) *is exactly controllable in time T.*

(ii) (1.1) *is null-controllable in time T.*

(iii) (1.1) *is approximately controllable in time T.*

Therefore, from now on, we shall only say "controllable in time T".

Proof. Since $\operatorname{Im} F_T = \mathbb{R}^n$, it is clear that (1.7) and (1.8) are equivalent. On the other hand, (1.7) and (1.9) are clearly equivalent since $\operatorname{Im} G_T$ is a finite dimensional subspace and therefore it is closed. □

Remark 1.1.6. We arbitrarily chose to consider controls which are in $L^2(0,T)^m$ but let us mention that we can actually consider any dense subspace of $L^2(0,T)^m$ as control set. Indeed, for any subspace $V \subset L^2(0,T)^m$, we have

$$\operatorname{Im} G_{T|\overline{V}} \subset \overline{\operatorname{Im} G_{T|V}} = \operatorname{Im} G_{T|V},$$

where the inclusion holds because G_T is continuous (see (1.3)) and the equality holds because $\operatorname{Im} G_{T|V}$ is finite dimensional. In particular, if there exists a control which is barely in $L^2(0,T)^m$, then there exists as well a control which is smooth, say in $C_c^\infty(0,T)^m$.

1.2 Duality

Since $G_T \in \mathcal{L}(L^2(0,T)^m, \mathbb{R}^n)$ thanks to (1.3), we have

$$\overline{\operatorname{Im} G_T} = \mathbb{R}^n \quad \Longleftrightarrow \quad \ker G_T^* = \{0\}\,. \tag{1.10}$$

Thus, we want compute G_T^*. To this end we introduce the so-called adjoint system of (1.1), that is

$$\left\{ \begin{array}{rcl} -\dfrac{d}{dt}z &=& A^*z, \quad t \in (0,T), \\ z(T) &=& z^1, \end{array} \right. \tag{1.11}$$

where $z^1 \in \mathbb{R}^n$. Then, multiplying (1.1) by z and integrating by parts we obtain the following fundamental relation:

$$y(T) \cdot z^1 - y^0 \cdot z(0) = \int_0^T u(t) \cdot B^* z(t)\, dt, \tag{1.12}$$

valid for every $y^0 \in \mathbb{R}^n$, $z^1 \in \mathbb{R}^n$ and $u \in L^2(0,T)^m$. In (1.12) and in the sequel, $\cdot$ denotes the inner product (in $\mathbb{R}^n$ or in $\mathbb{R}^m$). Thanks to (1.12) we readily see that

$$\begin{array}{rcccc} G_T^* & : & \mathbb{R}^n & \longrightarrow & L^2(0,T)^m \\ & & z^1 & \longmapsto & B^* z. \end{array} \tag{1.13}$$

Using (1.10), we have obtained the following fundamental result:

Theorem 1.2.1 (Duality). (1.1) *is controllable in time T if, and only if,*

$$\forall z^1 \in \mathbb{R}^n, \quad \left(B^* z(t) = 0, \quad \forall t \in [0,T]\right) \Longrightarrow z^1 = 0. \tag{1.14}$$

Remark 1.2.2. Clearly, (1.14) is equivalent to

$$\forall z^1 \in \mathbb{R}^n, \quad \left(B^* \tilde{z}(t) = 0, \quad \forall t \in [0,T]\right) \Longrightarrow z^1 = 0,$$

where $\tilde{z}(t) = z(T-t)$. But $\tilde{z}$ is analytic on $(0,+\infty)$. Thus, (1.14) holds if, and only if,

$$\forall z^1 \in \mathbb{R}^n, \quad \left(B^* \tilde{z}(t) = 0, \quad \forall t \in [0,+\infty)\right) \Longrightarrow z^1 = 0.$$

Therefore, the controllability of (1.1) does not depend on the time of control T. In other words, if there exists $T > 0$ such that (1.1) is controllable in time T, then, for every $T > 0$, (1.1) is controllable in time T. For this reason, in the sequel we shall only say that (1.1) is "controllable".

Remark 1.2.3. The strength of the duality is that it reduces the task of proving an existence result (existence of a control) to the task of proving a uniqueness result, which is often easier to handle.

1.3 Conditions of controllability

1.3.1 Gramian of controllability

Theorem 1.3.1. *Let $T > 0$. (1.1) is controllable if, and only if, the $n \times n$ matrix*

$$\Lambda_T = \int_0^T e^{(T-t)A} B B^* e^{(T-t)A^*} \, dt, \tag{1.15}$$

is invertible. Λ_T is called the Gramian of controllability or HUM operator.

Remark 1.3.2. Note that Λ_T is always symmetric and positive semi-definite. In particular, it is invertible if, and only if, it is positive definite. Now observe that Λ_T is positive definite if, and only if, there exists $C_T > 0$ such that

$$\left\|z^1\right\|^2 \leq C_T^2 \int_0^T \left\|B^* z(t)\right\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n. \tag{1.16}$$

This inequality is called observability inequality and the best constant $C_T > 0$ in (1.16) is called the control cost. We shall come back to this notion later on in Section 1.4.2.

Proof. By Theorem 1.2.1, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Clearly, this is equivalent to

$$\ker G_T G_T^* = \{0\}.$$

By definition of G_T (see (1.5)) and computation of G_T^* (see (1.13)) we readily see that $G_T G_T^* = \Lambda_T$. □

1.3.2 Kalman rank condition

Lemma 1.3.3. *For every $T > 0$, we have*

$$\ker G_T^* = \left(\operatorname{Im}\left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

Proof. By (1.13), $z^1 \in \ker G_T^*$ if, and only if,

$$B^* z(t) = 0, \quad \forall t \in [0, T], \tag{1.17}$$

where $z(t) = e^{(T-t)A^*} z^1$ is the solution to the adjoint system (1.11). Since z is analytic on $(0, T)$, we have (1.17) if, and only if, for some $0 < t_0 < T$,

$$\frac{d^k}{dt^k}(B^* z)(t_0) = 0, \quad \forall k \in \{0, 1, \ldots\}.$$

Computing $B^* z$ that gives

$$B^*(A^*)^k z^1 = 0, \quad \forall k \in \{0, 1, \ldots\}.$$

By the Cayley-Hamilton theorem, this is equivalent to

$$B^*(A^*)^k z^1 = 0, \quad \forall k \in \{0, 1, \ldots, n-1\}.$$

To summarize, $z^1 \in \ker G_T^*$ if, and only, if

$$z^1 \in \ker \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix}.$$

To conclude, observe that

$$\ker \begin{pmatrix} B^* \\ B^* A^* \\ \vdots \\ B^*(A^*)^{n-1} \end{pmatrix} = \ker \left(B|AB|\cdots|A^{n-1}B\right)^* = \left(\operatorname{Im}\left(B|AB|\cdots|A^{n-1}B\right)\right)^{\perp}.$$

□

An immediate consequence of Theorem 1.2.1 and Lemma 1.3.3 is the following fundamental result:

Theorem 1.3.4 (Kalman rank condition)**.** (1.1) *is controllable if, and only if,*

$$\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n. \tag{1.18}$$

Observe that, as expected (see Remark 1.2.2), the condition (1.18) does not depend on the time of control T.

The Kalman rank condition is an easy checkable condition for the controllability as it is shown on the following example.

Example 1.3.5. *The* 2×2 *system*

$$\begin{cases} \dfrac{d}{dt}y_1 = a_{11}y_1 + a_{12}y_2 + u, \\ \dfrac{d}{dt}y_2 = a_{21}y_1 + a_{22}y_2, \\ y_1(0) = y_1^0, \quad y_2(0) = y_2^0, \end{cases} \qquad t \in (0,T),$$

is controllable if, and only if,

$$a_{21} \neq 0.$$

Remark 1.3.6. Thanks to the Kalman rank condition we also see that we can fix the end-points of the control (and of its derivatives). Indeed, say that we look for controls u such that, in addition,

$$u(0) = u^0, \quad u(T) = u^1,$$

for some $u^0, u^1 \in \mathbb{R}^m$. Then, to this end we consider u as a new variable and we introduce the $(n+m)\times(n+m)$ augmented system

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, \\ \frac{d}{dt}u = v, \\ y(0) = y^0, \quad u(0) = u^0, \end{cases} \qquad t \in (0,T),$$

where v is now the control. We easily check that this system satisfies the associated Kalman rank condition.

Actually, we even have a stronger result than Theorem 1.3.4 since we can give a precise characterization of the reachable states:

Theorem 1.3.7. *Let* $y^0, y^1 \in \mathbb{R}^n$ *and* $T > 0$ *be fixed. There exists* $u \in L^2(0,T)^m$ *such that the corresponding solution* y *to* (1.1) *satisfies* $y(T) = y^1$ *if, and only if,*

$$y^1 - e^{TA}y^0 \in \operatorname{Im}\left(B|AB|\cdots|A^{n-1}B\right).$$

Proof. Using (1.6) we readily see that there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to (1.1) satisfies $y(T) = y^1$ if, and only if,

$$y^1 - e^{TA}y^0 \in \operatorname{Im} G_T.$$

Since $\operatorname{Im} G_T = (\ker G_T^*)^\perp$, the result follows from Lemma 1.3.3. □

There is a canonical form of controllable systems.

Proposition 1.3.8 (Canonical form of Brunovski)**.** *Assume that $m = 1$. Assume that* (1.18) *holds and let $K = (B|AB|\cdots|A^{n-1}B)$ (note that $K \in \mathbb{R}^{n\times n}$). Then,*

$$K^{-1}AK = \tilde{A}, \quad K^{-1}B = \tilde{B},$$

with

$$\tilde{A} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \alpha_0 \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{n-1} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \tag{1.19}$$

where $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$ are the coefficients of the characteristic polynomial of A, namely $p(\lambda) = \lambda^n - \alpha_{n-1}\lambda^{n-1} - \ldots - \alpha_0$, $\lambda \in \mathbb{C}$.

Proof. The proof is a simple computation of $K\tilde{A}$ and $K\tilde{B}$. □

It is worth mentioning that, once we know the "good" condition for the controllability (namely, (1.18)), there exists a direct proof of Theorem 1.3.4. By direct proof we mean a proof that is not using the duality at all. It is based on Proposition 1.3.8 and the following result, that we shall prove in a self-contained way:

Proposition 1.3.9. *Let $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{R}$. The system*

$$\begin{cases} \dfrac{d}{dt}y &= \tilde{A}y + \tilde{B}u, \quad t \in (0,T), \\ y(0) &= y^0, \end{cases} \tag{1.20}$$

where $\tilde{A}$ and $\tilde{B}$ are given by (1.19)*, is controllable.*

Remark 1.3.10. Combining Proposition 1.3.8 with 1.3.9 this gives a direct proof of the implication "$\Longleftarrow$" in Theorem 1.3.4 for $m = 1$.

Proof of Proposition 1.3.9 (without using Theorem 1.3.4). We give a direct proof. We recall that it is sufficient to only consider the target $y^1 = 0$ (see Proposition 1.1.5). Let $\overline{y}$ be the free solution to (1.20), that is the solution to (1.20) with $u = 0$. Let us introduce a cut-off function $\eta \in C^\infty([0,T])$ such that

$$\eta = 1 \text{ on } [0, T/3], \quad \eta = 0 \text{ on } [2T/3, T].$$

Observe that, because of the structure (1.19), the last equation of (1.20) is

$$\frac{d}{dt} y_n = y_{n-1} + \alpha_{n-1} y_n.$$

We set

$$y_n = \eta \overline{y}_n.$$

Then, we have no choice for y_{n-1} but to set

$$y_{n-1} = \frac{d}{dt} y_n - \alpha_{n-1} y_n.$$

By induction, we have to set

$$y_k = \frac{d}{dt} y_{k+1} - \alpha_k y_n, \quad \forall k \in \{n-2, \ldots, 1\},$$

and then

$$u = \frac{d}{dt} y_1 - \alpha_0 y_n.$$

Finally, thanks to the definition of η, note that

$$\forall k \in \{1, \ldots, n\}, \quad \begin{cases} y_k = \overline{y}_k & \text{on } [0, T/3], \\ y_k = 0 & \text{on } [2T/3, T], \end{cases}$$

so that

$$y(0) = y^0, \quad y(T) = 0.$$

□

1.3.3 Fattorini-Hautus test

There is another important characterization of the controllability, which is a dual version of the Kalman rank condition:

Theorem 1.3.11 (Fattorini-Hautus test)**.** (1.1) *is controllable if, and only if,*

$$\ker(\lambda - A^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{1.21}$$

Remark 1.3.12. Theorem 1.3.11 shows in particular that the following condition is necessary for the controllability:

$$\dim \ker(\lambda - A^*) \leq m, \quad \forall \lambda \in \mathbb{C}.$$

Indeed, assume that there exists a linearly independent family $\phi_1, \ldots, \phi_{m+1}$ of $\ker(\lambda - A^*)$. Then, $B^*\phi_1, \ldots, B^*\phi_{m+1}$ is linearly dependent as $B^* \in \mathbb{R}^{m \times n}$. Thus, there exists $(\alpha_1, \ldots, \alpha_{m+1}) \neq (0, \ldots, 0)$ such that $\sum_{k=1}^{m+1} \alpha_k B^* \phi_k = 0$. Let $z^1 = \sum_{k=1}^{m+1} \alpha_k \phi_k$. Then, $B^* z^1 = 0$. But $z^1 \in \ker(\lambda - A^*)$. Therefore, (1.21) implies that $z^1 = 0$, that is $\alpha_1 = \ldots = \alpha_{m+1} = 0$, a contradiction.

Proof. By Theorem 1.2.1, (1.1) is controllable if, and only if,

$$\ker G_T^* = \{0\}.$$

Assume that $\ker G_T^* = \{0\}$. Let $z^1 \in \ker(\lambda - A^*) \cap \ker B^*$. Then, $z(t) = e^{\lambda(T-t)} z^1$ and $B^* z(t) = e^{\lambda(T-t)} B^* z^1 = 0$ for every $t \in [0, T]$. Therefore, $z^1 = 0$ by assumption. Conversely, assume that $\ker G_T^* \neq \{0\}$. Let us first prove that:

(i) $\ker G_T^* \subset \ker B^*$.

(ii) $A^*(\ker G_T^*) \subset \ker G_T^*$.

Let $z^1 \in \ker G_T^*$. Then,

$$B^* z(t) = 0, \quad \forall t \in [0, T].$$

Taking $t = T$ we obtain $B^* z^1 = 0$, that is $z^1 \in \ker B^*$. On the other hand, taking the derivative we obtain

$$B^* e^{(T-t)A^*} A^* z^1 = 0, \quad \forall t \in [0, T],$$

that is $A^* z^1 \in \ker G_T^*$. Consequently, by (ii) we see the restriction of A^* to $\ker G_T^*$ is a linear operator from the finite dimensional space $\ker G_T^*$ into itself and, since $\ker G_T^* \neq \{0\}$, therefore possesses at least one complex eigenvalue. Since in addition by (i) we have $\ker G_T^* \subset \ker B^*$, this shows that there exist $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{R}^n$ with $\phi \neq 0$ such that

$$A^* \phi = \lambda \phi, \quad B^* \phi = 0.$$

This proves that (1.21) fails. □

1.3.4 Partial controllability

Sometimes we want to control not all but only some components of the system (1.1). This leads to the notion of partial controllability (also called output controllability in the literature).

Definition 1.3.13 (Partial controllability). Let $P \in \mathbb{R}^{p\times n}$ with $p \in \mathbb{N}^*$. We say that the system (1.1) is partially controllable if, for every $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^p$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.1) satisfies

$$Py(T) = y^1.$$

One can take for instance the projection on the first p components:

$$\begin{array}{cccc} P & : & \mathbb{R}^p \times \mathbb{R}^{n-p} & \longrightarrow & \mathbb{R}^p \\ & & (y_1, y_2) & \longmapsto & y_1, \end{array}$$

where $p \in \{1, \dots, n-1\}$ is the number of components we would like to control.

Mimicking the procedure developed in the previous sections, we see that (1.1) is partially controllable if, and only if,

$$\overline{\operatorname{Im} PG_T} = \mathbb{R}^n.$$

This is equivalent to

$$\ker G_T^* P^* = \{0\}.$$

Thanks to the expression of G_T^* (see (1.13)) we see that this is also equivalent to

$$\forall z^1 \in \mathbb{R}^p, \quad \Big(B^* z(t) = 0, \quad \text{a.e. } t \in (0,T)\Big) \Longrightarrow z^1 = 0,$$

where z is the solution to the following adjoint system:

$$\left\{ \begin{array}{rcl} -\dfrac{d}{dt} z & = & A^* z, \quad t \in (0,T), \\ z(T) & = & P^* z^1. \end{array} \right.$$

Reproducing the proof of Lemma 1.3.3 we easily obtain the following result:

Theorem 1.3.14 (Kalman rank condition). (1.1) *is partially controllable if, and only if,*

$$\operatorname{rank}(PB|PAB|\cdots|PA^{n-1}B) = p.$$

1.3.5 Higher order O.D.E.s

An interesting consequence of Theorem 1.3.4 is that it also gives a characterization of the controllability of higher order systems.

Let $y^0, \dot{y}^0 \in \mathbb{R}^n$ and let us consider the second order system:

$$\left\{ \begin{array}{l} \dfrac{d^2}{dt^2} y = Ay + Bu, \quad t \in (0,T), \\ y(0) = y^0, \quad \dfrac{d}{dt} y(0) = \dot{y}^0. \end{array} \right. \tag{1.22}$$

Firstly, we should point out that there are a priori several ways to define to controllability of (1.22). Do we want to achieve $y(T) = \frac{d}{dt}y(T) = 0$ or only $y(T) = 0$ for instance ? Note that the first goal is more physical since $(0,0) \in \mathbb{R}^n \times \mathbb{R}^n$ is a stable state of the system (1.22) (while $(0, \dot{y}^1)$ is not) and, therefore, once the system has reached this state, it stays at this state without any additional control required. However, we will study the two situations as both are of mathematical interest.

Definition 1.3.15 (Controllability)**.** We say that the system (1.22) is:

(i) controllable if, for every $y^0, \dot{y}^0 \in \mathbb{R}^n$ and $y^1, \dot{y}^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.22) satisfies

$$y(T) = y^1, \quad \frac{d}{dt}y(T) = \dot{y}^1.$$

(ii) partially controllable if, for every $y^0, \dot{y}^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)^m$ such that the corresponding solution y to system (1.22) satisfies

$$y(T) = y^1.$$

Of course the notion of partial controllability for (1.22) coincides with the notion of partial controllability of Section 1.3.4 for an underlying first order system. Surprisingly enough, it turns out that the notions of controllability and partial controllability for (1.22) are equivalent.

Theorem 1.3.16. *The following statements are equivalent:*

(i) (1.22) *is controllable.*

(ii) (1.22) *is partially controllable.*

(iii) $\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n.$

Proof. "(i) $\Longleftrightarrow$ (iii)". Introducing the new variable

$$\tilde{y} = \begin{pmatrix} y \\ \frac{d}{dt}y \end{pmatrix} \in \mathbb{R}^{2n},$$

and

$$\tilde{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ A & 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad \tilde{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathbb{R}^{2n\times m},$$

we see that the $n \times n$ second order system (1.22) is controllable if, and only if, so is the following $2n \times 2n$ first order system:

$$\begin{cases} \dfrac{d}{dt}\tilde{y} &= \tilde{A}\tilde{y} + \tilde{B}u, \quad t \in (0,T), \\ \tilde{y}(0) &= \tilde{y}^0. \end{cases} \tag{1.23}$$

By Theorem 1.3.4, the controllability of (1.23) is equivalent to the corresponding Kalman rank condition, that is

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2n.$$

A computation shows that

$$(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = \begin{pmatrix} 0 & B & 0 & AB & \cdots & 0 & A^{n-1}B \\ B & 0 & AB & 0 & \cdots & A^{n-1}B & 0 \end{pmatrix}. \tag{1.24}$$

Therefore,

$$\operatorname{rank}(\tilde{B}|\tilde{A}\tilde{B}|\cdots|\tilde{A}^{2n-1}\tilde{B}) = 2\operatorname{rank}(B|AB|\cdots|A^{n-1}B).$$

"(ii) $\Longleftrightarrow$ (iii)". Let us introduce

$$\begin{array}{rccl} P & : & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & & (y_1, y_2) & \longmapsto & y_1. \end{array}$$

Then, the $n \times n$ second order system (1.22) is partially controllable if, and only if, the $2n \times 2n$ first order system (1.23) is partially controllable. By Theorem 1.3.14, this is equivalent to the corresponding Kalman rank condition, namely

$$\operatorname{rank}(P\tilde{B}|P\tilde{A}\tilde{B}|\cdots|P\tilde{A}^{2n-1}\tilde{B}) = n.$$

Thanks to (1.24) we readily see that it is equivalent to $\operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n$. □

1.4 Optimal controls

1.4.1 Control of minimal L^2-norm

Assume that (1.1) is controllable. A priori there is no reason for a control to be unique. Let $y^0, y^1 \in \mathbb{R}^n$ and let us introduce the corresponding set of controls

$$U = \left\{ u \in L^2(0,T)^m, \quad y(T) = y^1 \right\}.$$

We consider the minimization problem

$$\min_{u \in U} \frac{1}{2} \|u\|^2_{L^2(0,T)^m}.$$

A solution of this problem will be called a control of minimal L^2-norm.

Theorem 1.4.1 (L^2-optimal control)**.** *Assume that* (1.1) *is controllable. Then, for every* $y^0, y^1 \in \mathbb{R}^n$*, there exists a unique control of minimal* L^2*- norm and it is given by*

$$u_{\text{opt}}(t) = B^* e^{(T-t)A^*} \Lambda_T^{-1} \left(y^1 - e^{TA} y^0 \right), \tag{1.25}$$

where $\Lambda_T \in \mathbb{R}^{n \times n}$ *is the Gramian of controllability (see* (1.15)*). The control* u_{opt} *is also called the HUM control.*

Remark 1.4.2. Note that the control u_{opt} is analytic on $\mathbb{R}$.

Lemma 1.4.3 (Hilbert projection theorem)**.** *Let H be a Hilbert space. Let $C \subset H$ be a nonempty closed convex. For every $x \in H$, there exists a unique $p \in C$ such that*

$$\|x - p\| = \min_{y \in C} \|x - y\|.$$

Moreover, p is the unique element of C that satisfies

$$\langle x - p, y - p \rangle \leq 0, \quad \forall y \in C.$$

Proof of Theorem 1.4.1. Firstly, observe that U is not empty by assumption. Let then $u_0 \in U$. We easily see that

$$U = u_0 + \ker G_T.$$

Therefore, U is an affine subspace of $\mathbb{R}^n$. In particular, it is a closed convex and, by Lemma 1.4.3, there exists a unique $u_{\text{opt}} \in U$ (the projection of 0 on U) such that

$$\|u_{\text{opt}}\| = \min_{u \in U} \|u\|.$$

Moreover, we have

$$\langle u_{\text{opt}}, u_{\text{opt}} - u \rangle_{L^2} \leq 0, \quad \forall u \in U.$$

Since $U = u_{\text{opt}} + \ker G_T$, this gives

$$\langle u_{\text{opt}}, v \rangle_{L^2} = 0, \quad \forall v \in \ker G_T.$$

Thus,

$$u_{\text{opt}} \in (\ker G_T)^{\perp} = \overline{\operatorname{Im} G_T^*}.$$

Then, there exists $(z_n^1)_n$ such that

$$G_T^* z_n^1 \xrightarrow[n \to +\infty]{} u_{\text{opt}}. \tag{1.26}$$

But $(z_n^1)_n$ converges. Indeed, since $u_{\text{opt}} \in U$, we have

$$G_T u_{\text{opt}} = y^1 - F_T y^0.$$

Since G_T is a bounded operator, combined with (1.26) this gives

$$G_T G_T^* z_n^1 \xrightarrow[n \to +\infty]{} y^1 - F_T y^0.$$

By Theorem 1.15, we know that $G_T G_T^* = \Lambda_T$ is invertible. Thus,

$$z_n^1 \xrightarrow[n \to +\infty]{} \Lambda_T^{-1} \left(y^1 - F_T y^0 \right).$$

Coming back to (1.26), we obtain that

$$u_{\text{opt}} = G_T^* \Lambda_T^{-1} \left(y^1 - F_T y^0 \right).$$

The expressions of G_T^* (see (1.13)) and F_T finally give (1.25). $\square$

1.4.2 Control cost

In section we consider $y^0 = 0$. Assume that (1.1) is controllable. Then, the map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & L^2(0,T)^m \\ y^1 & \longmapsto & u_{\text{opt}}, \end{array}$$

is a bounded linear map (see for instance (1.25)). We denote by C_T its norm of operator.

Definition 1.4.4 (Control cost)**.** Assume that (1.1) is controllable. Then, the quantity

$$C_T = \sup_{\substack{y^1 \in \mathbb{R}^n \\ y^1 \neq 0}} \frac{\|u_{\text{opt}}\|_{L^2(0,T)^m}}{\|y^1\|} = \sup_{\substack{y^1 \in \mathbb{R}^n \\ \|y^1\|=1}} \|u_{\text{opt}}\|_{L^2(0,T)^m}, \tag{1.27}$$

where u_{opt} is the control of minimal L^2-norm steering the solution y to (1.1) from $y^0 = 0$ to y^1 in time T, is called the control cost.

The following proposition gives a dual characterization for the control cost:

Proposition 1.4.5. *Assume that* (1.1) *is controllable. The control cost C_T satisfies*

$$C_T = \sup_{\substack{z^1 \in \mathbb{R}^n \\ z^1 \neq 0}} \frac{\|z^1\|}{\sqrt{\int_0^T \|B^* z(t)\|^2 \, dt}} = \sup_{\substack{z^1 \in \mathbb{R}^n \\ \|z^1\|=1}} \frac{1}{\sqrt{\int_0^T \|B^* z(t)\|^2 \, dt}}, \tag{1.28}$$

where z is the solution to the adjoint system (1.11)*. In other words, the control cost C_T is the best constant $C > 0$ such that the following inequality (called observability inequality) holds:*

$$\|z^1\|^2 \leq C^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n.$$

Remark 1.4.6. Since the closed unit ball is compact in $\mathbb{R}^n$, both supremum in (1.27) and in (1.28) are actually maximum.

Proof. By homogeneity the second equality in (1.28) is clear. Next, observe that (see (1.15))

$$\int_0^T \|B^* z(t)\|^2 \, dt = \Lambda_T z^1 \cdot z^1, \quad \forall z^1 \in \mathbb{R}^n,$$

and $\Lambda_T z^1 \cdot z^1 \neq 0$ for every $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ by controllability. Let $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ be fixed. Let $y^1 = z^1$ and let u_{opt} be the associated optimal control. Using (1.12) and the Cauchy-Schwarz inequality we have

$$\|z^1\|^2 \leq \left(\int_0^T \|u_{\text{opt}}(t)\|^2 \, dt \right)^{\frac{1}{2}} \left(\Lambda_T z^1 \cdot z^1 \right)^{\frac{1}{2}}.$$

It follows that

$$\frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1} \leq \frac{\int_0^T \left\|u_{\mathrm{opt}}(t)\right\|^2 \, dt}{\|z^1\|^2} = \frac{\int_0^T \left\|u_{\mathrm{opt}}(t)\right\|^2 \, dt}{\|y^1\|^2}.$$

This shows that the supremum is finite with

$$\sup_{\substack{z^1 \in \mathbb{R}^n \\ z^1 \neq 0}} \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1} \leq C_T^2.$$

Conversely, let $y^1 \in \mathbb{R}^n$ with $y^1 \neq 0$ and let u_{opt} be the associated optimal control. Set

$$z^1 = \Lambda_T^{-1} y^1.$$

Using (1.12) and the expression (1.25) of u_{opt} we obtain

$$\Lambda_T z^1 \cdot z^1 = \int_0^T \left\|u_{\mathrm{opt}}(t)\right\|^2 \, dt.$$

Thus,

$$\frac{\int_0^T \left\|u_{\mathrm{opt}}(t)\right\|^2 \, dt}{\|y^1\|^2} = \frac{\int_0^T \left\|u_{\mathrm{opt}}(t)\right\|^2 \, dt}{\|\Lambda_T z^1\|^2} = \frac{\Lambda_T z^1 \cdot z^1}{\|\Lambda_T z^1\|^2}.$$

But

$$\frac{\Lambda_T z^1 \cdot z^1}{\|\Lambda_T z^1\|^2} = \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1} \frac{\left|\Lambda_T z^1 \cdot z^1\right|^2}{\|\Lambda_T z^1\|^2 \, \|z^1\|^2} \leq \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1}.$$

This establishes the reversed inequality

$$C_T^2 \leq \sup_{\substack{z^1 \in \mathbb{R}^n \\ z^1 \neq 0}} \frac{\left\|z^1\right\|^2}{\Lambda_T z^1 \cdot z^1}.$$

□

Proposition 1.4.7. *Assume that* (1.1) *is controllable. The control cost* C_T *satisfies:*

(i) $C_T \to +\infty$ *as* $T \to 0^+$.

(ii) C_T *is decreasing.*

Proof. By Proposition 1.4.5 we have

$$C_T = \sup_{\substack{z^1 \in \mathbb{R}^n \\ \left\|z^1\right\| = 1}} \frac{1}{\sqrt{\int_0^T \left\|B^* \tilde{z}(t)\right\|^2 \, dt}},$$

where $\tilde{z}(t) = z(T-t)$ does not depend on T. Then,

$$C_T \geq \frac{1}{\sqrt{\int_0^T \|B^*\tilde{z}(t)\|^2 \, dt}} \xrightarrow[T \to 0^+]{} +\infty.$$

To prove the second point, we simply observe that, for every $T' \geq T$, we have

$$\int_0^{T'} \|B^*\tilde{z}(t)\|^2 \, dt \geq \int_0^{T} \|B^*\tilde{z}(t)\|^2 \, dt, \tag{1.29}$$

from which it immediately follows that

$$C_{T'} \leq C_T, \quad \forall T' \geq T.$$

□

Remark 1.4.8. Using Remark 1.4.6 we see that C_T is actually strictly decreasing. Indeed, the inequality (1.29) is strict for $T' > T$ because we can not have $B^*\tilde{z}(t) = 0$ for $t \in [T, T']$ by controllability. Taking the inverse and then the maximum over all $z^1 \in \mathbb{R}^n$ with $\|z^1\| = 1$ we obtain that $C_{T'} < C_T$.

Remark 1.4.9. Since C_T is decreasing and bounded from below by 0, we have $C_T \to \inf_{T>0} C_T$ as $T \to +\infty$. However, it is not true that $\inf_{T>0} C_T = 0$ in general. Indeed, assume for instance that A has an unstable eigenvalue $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$. Then, taking $z^1 = \phi$, where ϕ is a normalized eigenvector of A^* associated with $\bar{\lambda}$, a computation gives

$$C_T^2 \geq \frac{-2\operatorname{Re}\lambda}{\|B^*\phi\|^2}, \quad \forall T > 0.$$

Therefore $\inf_{T>0} C_T > 0$. This feature can be explained by remarking that, on the one hand the system naturally dissipates to 0 in the direction of ϕ but on the other hand, the goal is to reach a state that can be different from 0. Of course, this also happens because we deal with the notion of exact controllability.

Let us conclude this section by mentioning that we can actually obtain a very precise asymptotic of the control cost as $T \to 0^+$ (the proof is admitted, see e.g. [Sei88]).

Theorem 1.4.10 (Estimate of the control cost)**.** *Assume that* (1.1) *is controllable and let* $r \in \{0, \ldots, n-1\}$ *be the smallest exponent such that* $\operatorname{rank}(B|AB|\cdots|A^rB) = n$. *Then, there exists* $\gamma > 0$ *such that*

$$C_T \sim \frac{\gamma}{T^{r+\frac{1}{2}}} \quad \text{as } T \to 0^+.$$

1.4.3 Variational approach

In this section we provide another approach to look at the optimal control problem. Let us go back to the fundamental identity (1.12) with $y^1 = 0$. We readily see that $y(T) = 0$ if, and only if,

$$0 = \int_0^T u(t) \cdot B^* z(t)\, dt + y^0 \cdot z(0), \quad \forall z^1 \in \mathbb{R}^n, \tag{1.30}$$

This identity can be viewed an optimality condition for the extremal points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$J(z^1) = \frac{1}{2} \int_0^T \|B^* z(t)\|^2 \, dt + y^0 \cdot z(0),$$

where z is the solution to the adjoint system (1.11).

Theorem 1.4.11. *Assume that the system* (1.1) *is controllable. Then, for every $y^0 \in \mathbb{R}^n$, J has a minimizer. Moreover, if z^1_{opt} is a minimizer of J and z_{opt} denotes the corresponding solution to the adjoint system* (1.11)*, then, the solution y to* (1.1) *corresponding to*

$$u_{\text{opt}} = B^* z_{\text{opt}},$$

satisfies $y(T) = 0$. Moreover, u_{opt} is the unique null-control of minimal L^2-norm.

Lemma 1.4.12. *Let $J : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous and convex function that is also coercive, that is*

$$J(z^1) \xrightarrow[\|z^1\| \to +\infty]{} +\infty.$$

Then, J has (at least one) minimizer.

Proof of Theorem 1.4.11. Clearly, J is continuous and convex on $\mathbb{R}^n$. Let us show that it is coercive. Let us introduce the function $N : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ defined by

$$N(z^1) = \int_0^T \|B^* z(t)\|^2 \, dt, \quad z^1 \in \mathbb{R}^n,$$

where z is the solution to the adjoint system (1.11). Since (1.1) is controllable, N defines a norm on $\mathbb{R}^n$. Since all the norms are equivalent in finite dimension, there exists $C > 0$ such that

$$\left\|z^1\right\|^2 \leq C^2 \int_0^T \|B^* z(t)\|^2 \, dt, \quad \forall z^1 \in \mathbb{R}^n.$$

It follows that

$$\begin{aligned} J(z^1) &\geq \frac{1}{2C^2} \left\|z^1\right\|^2 - \left|y^0 \cdot z(0)\right| \\ &\geq \frac{1}{2C^2} \left\|z^1\right\|^2 - \alpha \left\|z^1\right\|, \end{aligned}$$

with $\alpha = \|y^0\| e^{T\|A^*\|}$. Therefore,

$$J(z^1) \xrightarrow[\|z^1\| \to +\infty]{} +\infty.$$

By Lemma 1.4.12, J has a minimizer z^1_{opt}. Next, note that J is differentiable on $\mathbb{R}^n$ with

$$DJ(\hat{z}^1)z^1 = \int_0^T B^*\hat{z}(t) \cdot B^*z(t)\,dt + y^0 \cdot z(0), \quad \forall \hat{z}^1, z^1 \in \mathbb{R}^n,$$

where z (*resp.* $\hat{z}$) is the solution to the adjoint system (1.11) associated with z^1 (*resp.* $\hat{z}^1$). Since z^1_{opt} is a minimizer of J, we have $DJ(z^1_{\text{opt}}) = 0$, that is

$$0 = \int_0^T B^*z_{\text{opt}}(t) \cdot B^*z(t)\,dt + y^0 \cdot z(0), \quad \forall z^1 \in \mathbb{R}^n.$$

This means that $u_{\text{opt}} = B^*z_{\text{opt}}$ is a null-control (see (1.30)). Let us finally prove that u_{opt} is the unique null-control of minimal L^2-norm. Let $u \in L^2(0,T)^m$ be another null-control. Since u and u_{opt} are two null-controls, they both satisfy (1.30). Taking $z^1 = z^1_{\text{opt}}$ in (1.30), we obtain

$$\int_0^T (u(t) - u_{\text{opt}}(t)) \cdot u_{\text{opt}}(t)\,dt = 0.$$

It follows that

$$\|u\|^2_{L^2(0,T)^m} = \|u_{\text{opt}}\|^2_{L^2(0,T)^m} + \|u - u_{\text{opt}}\|^2_{L^2(0,T)^m}.$$

From this identity we see that u_{opt} minimizes the L^2-norm among all possible null-controls and that it is the only one. □

1.5 Controls with constraints

In this section we will look for controls $u \in L^2(0,T)^m$ that satisfy in addition the constraint

$$u(t) \in U \quad \text{a.e. } t \in (0,T), \tag{1.31}$$

where U is a fixed nonempty subset of $\mathbb{R}^m$. Let us first point out that we have already encountered controls that satisfy some constraints, see Remarks 1.1.6 and 1.3.6. In this section we provide some elements of the general theory for systems with constrained controls.

1.5.1 Sufficient conditions for large times

Definition 1.5.1. Let $C \subset \mathbb{R}^n$ be the set of elements $y^0 \in \mathbb{R}^n$ such that there exist $T > 0$ and $u \in L^2(0,T)^m$ with (1.31) such that the corresponding solution y to (1.1) satisfies $y(T) = 0$. We say that the constrained system (1.1)-(1.31) is:

(i) null-controllable if $C = \mathbb{R}^n$.

(ii) locally-null controllable if $0 \in \mathring{C}$, where $\mathring{C}$ denotes the interior of the set C.

Remark 1.5.2. Observe that the time of control depends on the initial data in these definitions.

Let us start by investigating what the controllability of the unconstrained system (1.1) implies for the controllability of the constrained system (1.1)-(1.31).

Theorem 1.5.3. *Assume that $0 \in \mathring{U}$. The following statements are equivalent:*

(i) The system (1.1) *is controllable.*

(ii) The system (1.1)-(1.31) *is locally null-controllable.*

Proof. (i) $\Longrightarrow$ (ii). Assume that (1.1) is controllable. Then, by Theorem 1.4.1, there exists a control u_{opt} with

$$\|u_{\text{opt}}(t)\| \leq M \left\|y^0\right\|, \quad \forall t \in [0,T],$$

for some $M > 0$ that depends only on A, B and T. Since $0 \in \mathring{U}$ by assumption, there exists $r > 0$ such that, for every $u \in \mathbb{R}^m$, if $\|u\| < r$ then $u \in U$. Therefore, if y^0 is small enough, say $\left\|y^0\right\| < r/M$, then $u_{\text{opt}}(t) \in U$ for every $t \in [0,T]$ and $0 \in \mathring{C}$.

(ii) $\Longrightarrow$ (i). Conversely, assume that (1.1) is not controllable, that is

$$\operatorname{rank}(B|AB|\cdots|A^{n-1}B) < n.$$

Thus, there exists a non zero vector $\xi \in \mathbb{R}^n$ such that

$$\xi^* A^k B = 0, \quad \forall k \in \{0,1,\ldots,n-1\}.$$

Using the Cayley-Hamilton theorem it follows that

$$\xi^* A^k B = 0, \quad \forall k \in \{0,1,\ldots\}.$$

Thus,

$$\xi^* e^{tA} B = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \xi^* A^k B = 0, \quad \forall t \in \mathbb{R}.$$

Let now $y^0 \in C$. By definition, there exist $T > 0$ and $u \in L^2(0,T)^m$ such that

$$0 = e^{TA} y^0 + \int_0^T e^{(T-t)A} B u(t)\, dt,$$

or, equivalently,

$$0 = y^0 + \int_0^T e^{-tA} B u(t)\, dt.$$

Taking the inner product of this identity with ξ we obtain that

$$\xi \cdot y^0 = 0.$$

Since this is true for any $y^0 \in C$, this shows that $C \subset \xi^\perp$. But $\xi^\perp$ is a vectorial space, which is not $\mathbb{R}^n$ since $\xi \neq 0$, and therefore its interior is empty. It follows that C has an empty interior as well, so that $0 \notin \mathring{C}$. □

Let us now give an easy but interesting sufficient condition for the null-controllability of (1.1)-(1.31).

Theorem 1.5.4. *Assume that* $0 \in \mathring{U}$ *and:*

(i) System (1.1) *is controllable.*

(ii) A is stable (that is, $e^{tA}y^0 \to 0$ *as* $t \to +\infty$ *for every* $y^0 \in \mathbb{R}^n$*).*

Then, the system (1.1)-(1.31) *is null-controllable.*

Proof. By Theorem 1.5.3 we have $0 \in \mathring{C}$. Thus, there exists $r > 0$ such that, for every $y^0 \in \mathbb{R}^n$, if $\left\|y^0\right\| < r$, then $y^0 \in C$. Let $y^0 \in \mathbb{R}^n$ be fixed. Since A is stable, we have

$$e^{tA}y^0 \xrightarrow[t \to +\infty]{} 0.$$

Therefore, there exists $T_1 > 0$ (large enough and depending on y^0) such that

$$\left\|e^{T_1 A}y^0\right\| < r.$$

It follows that $e^{T_1 A}y^0 \in C$. By definition of C, there exist $T_2 > 0$ and $u_2 \in L^2(T_1, T_1+T_2)^m$, with $u_2(t) \in U$ for a.e. $t \in (T_1, T_1 + T_2)$, such that the solution y_2 to

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}y_2 & = & Ay_2 + Bu_2, \quad t \in (T_1, T_1 + T_2), \\ y_2(T_1) & = & e^{T_1 A}y^0, \end{array} \right.$$

satisfies $y_2(T_1 + T_2) = 0$. Thus, we see that the control defined by

$$u(t) = \left\{ \begin{array}{ll} 0 & \text{for } t \in (0, T_1), \\ u_2(t) & \text{for } t \in (T_1, T_1 + T_2), \end{array} \right.$$

satisfies (1.31) and brings the corresponding solution to (1.1) from y^0 to 0 in time T_1+T_2. □

In the case of bounded control sets, there is a complete characterization of the null-controllability (the proof is more complex though, see e.g. [Son98, Theorem 6] (applied to $-A$ and $-B$ instead of A and B)):

Theorem 1.5.5. *Assume that* $0 \in \mathring{U}$ *and that* U *is bounded. Then, the system* (1.1)-(1.31) *is null-controllable if, and only if, the following two conditions hold:*

(i) The system (1.1) *is controllable.*

(ii) $\operatorname{Re}\lambda \leq 0$ *for every eigenvalue* $\lambda \in \mathbb{C}$ *of* A*.*

We recall that A is stable if, and only if, $\operatorname{Re}\lambda < 0$ for every eigenvalue $\lambda \in \mathbb{C}$ of A (see e.g. [Zab08, Theorem I.2.3]). Therefore the condition (ii) of Theorem 1.5.4 is stronger than the condition (ii) of Theorem 1.5.5.

1.5.2 Time-optimal problems

In the previous section we provided some sufficient conditions to ensure the null-controllability of (1.1)-(1.31) for large enough times. Therefore, it is natural to address the problem of finding the best time possible and a possible corresponding control.

1.5.2.1 Existence of time-optimal controls

Definition 1.5.6 (Reachable set)**.** For $y^0 \in \mathbb{R}^n$ and $T > 0$, let $R_T(y^0) \subset \mathbb{R}^n$ be the set of elements $y^1 \in \mathbb{R}^n$ such that there exists $u \in L^2(0,T)^m$ with (1.31) such that the corresponding solution y to (1.1) satisfies $y(T) = y^1$. According to (1.2) it is the set of all elements

$$e^{TA}y^0 + \int_0^T e^{(T-t)A}Bu(t)\,dt, \tag{1.32}$$

for $u \in L^2(0,T)^m$ with (1.31). For $T = 0$ we naturally set $R_0(y^0) = \left\{y^0\right\}$.

Proposition 1.5.7 (Properties of the reachable set)**.** *Assume that* U *is compact. Let* $y^0 \in \mathbb{R}^n$ *be fixed. Then,*

(i) $R_T(y^0)$ *is compact and convex for every* $T \geq 0$*.*

(ii) $R_T(y^0)$ *varies continuously with respect to* T*. More precisely, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, for every* $T_1, T_2 \geq 0$*, if* $|T_1 - T_2| < \delta$ *then*

$$d(R_{T_1}(y^0), R_{T_2}(y^0)) \leq \varepsilon,$$

where $d(X_1, X_2)$ *denotes the Hausdorff distance between the closed subsets* $X_1 \subset \mathbb{R}^n$ *and* $X_2 \subset \mathbb{R}^n$*, that is* $d(X_1, X_2) = \max\left\{\sup_{x_1 \in X_1} d(x_1, X_2), \sup_{x_2 \in X_2} d(X_1, x_2)\right\}$.

For a proof of this proposition we refer to [LM67, Theorem 2.1] if U is convex and [LM67, Theorem 2.1A] for the general case.

Theorem 1.5.8 (Existence of time-optimal controls)**.** *Assume that U is compact. Let $y^0, y^1 \in \mathbb{R}^n$ be fixed. Assume that there exists $T \geq 0$ such that $y^1 \in R_T(y^0)$. Then, the set $\left\{T \geq 0,\, y^1 \in R_T(y^0)\right\}$ has a minimum $T_{\text{min}} \geq 0$. By definition, this means that $T_{\text{min}} = 0$ if, and only if, $y^1 = y^0$ and, if $T_{\text{min}} > 0$, this means that there exists $u \in L^2(0, T_{\text{min}})^m$ with* (1.31) *such that the corresponding solution y to* (1.1) *satisfies $y(T_{\text{min}}) = y^1$. Such a u is called a time-optimal control.*

Proof. Let

$$E = \left\{T \geq 0,\, y^1 \in R_T(y^0)\right\}.$$

By assumption, E is not empty. To prove that E has a minimum we show that it is closed. Let then $T_k \in E$, $k \in \mathbb{N}$, and $T \in \mathbb{R}$ be such that $T_k \to T$ as $k \to +\infty$. We have to prove that $T \in E$. Clearly, $T \geq 0$. Let us now prove that $y^1 \in R_T(y^0)$. Since $R_T(y^0)$ is closed (see Proposition 1.5.7), it is equivalent to prove that $d(y^1, R_T(y^0)) = 0$. Let $\varepsilon > 0$. We have

$$d(y^1, R_T(y^0)) \leq d(y^1, R_{T_k}(y^0)) + d(R_{T_k}(y^0), R_T(y^0)).$$

Since $y^1 \in R_{T_k}(y^0)$ by definition of T_k, we have $d(y^1, R_{T_k}(y^0)) = 0$. Now, since $T_k \to T$, by continuity (see Proposition 1.5.7) there exists $k \in \mathbb{N}$ large enough so that $d(R_{T_k}(y^0), R_T(y^0)) \leq \varepsilon$. Therefore, we have proved that $d(y^1, R_T(y^0)) \leq \varepsilon$ for every $\varepsilon > 0$, that is $d(y^1, R_T(y^0)) = 0$. □

1.5.2.2 Maximum principle

Before proving the so-called Pontryagin maximum principle, we establish some properties of time-optimal controls.

Definition 1.5.9 (Extremal control)**.** Let $y^0 \in \mathbb{R}^n$ and $T > 0$ be fixed. A function $u \in L^2(0, T)^m$ is called an extremal control if u satisfies (1.31) and the corresponding solution y to (1.1) satisfies $y(T) \in \partial R_T(y^0)$.

Theorem 1.5.10 (Time-optimal controls are extremal)**.** *Assume that U is compact. Let $y^0, y^1 \in \mathbb{R}^n$ be such that $y^1 \neq y^0$. Assume that there exists $T > 0$ such that $y^1 \in R_T(y^0)$. Let $T_{\text{min}} > 0$ be the optimal time and let $u \in L^2(0, T_{\text{min}})^m$ be a time-optimal control (whose existences are guaranteed by Theorem 1.5.8). Then, u is an extremal control.*

We will need the following result from convex analysis (for a proof, see e.g. [Zab08, Theorem III.3.5])

Lemma 1.5.11 (Hyperplane separation theorem)**.** *Let $C \subset \mathbb{R}^n$ be a convex subset and $a \in \mathbb{R}^n$. There exists $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ such that*

$$\xi \cdot y \leq \xi \cdot a, \quad \forall y \in C$$

if, and only if, $a \notin \mathring{C}$.

Proof of Theorem 1.5.10. We have to show that $y^1 \in \partial R_{T_{\min}}(y^0)$. Since $R_{T_{\min}}(y^0)$ is a closed convex (see Proposition 1.5.7), by Lemma 1.5.11, it is equivalent to prove that there exists $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ such that, for every $\hat{y}^1 \in R_{T_{\min}}(y^0)$,

$$\xi \cdot \hat{y}^1 \leq \xi \cdot y^1. \tag{1.33}$$

Let $T_k > 0$, $k \in \mathbb{N}^*$, be such that $T_k \to T_{\min}$ as $k \to +\infty$ with $T_k < T_{\min}$ for every $k \in \mathbb{N}^*$. Since $T_k < T_{\min}$, by definition of $T_{\min}$ we have

$$y^1 \notin R_{T_k}(y^0), \quad \forall k \in \mathbb{N}^*.$$

In particular $y^1 \notin \overset{\circ}{R_{T_k}}(y^0)$. Since $R_{T_k}(y^0)$ is convex, by Lemma 1.5.11 there exists $\xi_k \in \mathbb{R}^n$ with $\xi_k \neq 0$ such that, for every $w^1 \in R_{T_k}(y^0)$,

$$\xi_k \cdot w^1 \leq \xi_k \cdot y^1. \tag{1.34}$$

Since $\xi_k \neq 0$, we can assume that $\|\xi_k\| = 1$. Since $(\xi_k)_k$ is now a bounded sequence, we can extract a subsequence (still denoted by $(\xi_k)_k$) such that $\xi_k \to \xi$ as $k \to +\infty$ for some $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ (as $\|\xi\| = 1$). Let $\hat{y}^1 \in R_{T_{\min}}(y^0)$ be fixed. Take a sequence $(\hat{y}^1_j)_j$ such that $\hat{y}^1_j \to \hat{y}^1$ as $j \to +\infty$ with $\hat{y}^1_j \in R_{T_{k_j}}(y^0)$ for every $j \in \mathbb{N}^*$ for some $k_j \to +\infty$ as $j \to +\infty$. Such a sequence exists because of the continuity of the reachable sets (see Proposition 1.5.7). Indeed, since $T_j \to T_{\min}$ as $j \to +\infty$, for j large enough there exists k_j, with $k_j \to +\infty$ as $j \to +\infty$, such that

$$d(R_{T_{k_j}}(y^0), \hat{y}^1) < \frac{1}{j}.$$

Therefore, there exists $\hat{y}^1_j \in R_{T_{k_j}}(y^0)$ such that

$$d(\hat{y}^1_j, \hat{y}^1) < \frac{1}{j}.$$

Finally, taking $w^1 = \hat{y}^1_j$ in (1.34) and passing to the limit as $j \to +\infty$, we obtain (1.33). □

Thanks to Theorem 1.5.10 we can now focus on the notion of extremal control.

Theorem 1.5.12 (Pontryagin maximum principle)**.** *Assume that U is compact. Let $y^0 \in \mathbb{R}^n$, $T > 0$ and $u \in L^2(0,T)^m$. The following statements are equivalent:*

(i) u is an extremal control.

(ii) There exists $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ such that the corresponding solution z to the adjoint system

$$\begin{cases} -\dfrac{d}{dt}z = A^*z, & t \in (0,T), \\ z(T) = z^1, \end{cases}$$

satisfies

$$B^*z(t) \cdot u(t) = \max_{u \in U} B^*z(t) \cdot u \quad \text{a.e. } t \in (0,T). \tag{1.35}$$

Proof. By definition, u is an extremal control if, and only if, $y(T) \in \partial R_T(y^0)$, where y is the corresponding solution to (1.1). Since $R_T(y^0)$ is a closed convex (see Proposition 1.5.7), by Lemma 1.5.11 this is equivalent to the existence of $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ such that, for every $\hat{y}^1 \in R_T(y^0)$,

$$z^1 \cdot \hat{y}^1 \leq z^1 \cdot y(T).$$

Recalling (1.32), this inequality is equivalent to

$$z^1 \cdot \int_0^T e^{(T-t)A} B\left(\hat{u}(t) - u(t)\right) dt \leq 0,$$

that is,

$$\int_0^T B^* z(t) \cdot \left(\hat{u}(t) - u(t)\right) dt \leq 0, \tag{1.36}$$

for every $\hat{u} \in L^2(0,T)^m$ with $\hat{u}(t) \in U$ for a.e. $t \in (0,T)$. Therefore, if u satisfies (1.35) then, in particular,

$$B^* z(t) \cdot u(t) \geq B^* z(t) \cdot \hat{u}(t) \quad \text{a.e. } t \in (0,T),$$

for every $\hat{u} \in L^2(0,T)^m$ with $\hat{u}(t) \in U$ for a.e. $t \in (0,T)$. Integrating this inequality, we obtain (1.36). Conversely, assume that (1.36) holds and let us prove that u satisfies (1.35). It is clear that there exists a function $w : (0,T) \to U$ such that

$$\max_{u \in U} B^* z(t) \cdot u = B^* z(t) \cdot w(t), \quad \text{a.e. } t \in (0,T).$$

It can be proved that w can even be chosen so that $w \in L^2(0,T)^m$ (see e.g. [LM67, Lemma 1.2A and 1.3A]). In particular,

$$B^* z(t) \cdot w(t) \geq B^* z(t) \cdot u(t), \quad \text{a.e. } t \in (0,T),$$

and we can integrate this inequality to obtain the reverse inequality of (1.36) for $\hat{u} = w$. As a result, $t \longmapsto B^* z(t) \cdot w(t) - B^* z(t) \cdot u(t)$ is a positive function whose integral is zero and therefore is itself equal to zero. □

1.5.2.3 Bang-bang controls

Theorem 1.5.13 (Bang-bang principle). *Assume that U is compact. Let $y^0 \in \mathbb{R}^n$, $T > 0$ and $u \in L^2(0,T)^m$ Assume that (1.1) is controllable. If u is an extremal control, then*

$$u(t) \in \partial U, \quad \text{a.e. } t \in (0,T).$$

Lemma 1.5.14. *Let U be a closed subset of $\mathbb{R}^n$. Let $q \in \mathbb{R}^m$ and define the function $f : U \longrightarrow \mathbb{R}$ by $f(u) = q \cdot u$. Assume that $q \neq 0$. If $u_0 \in U$ is a point of local maximum of f, then $u_0 \in \partial U$.*

Proof. Let $u_0 \in U$ be a point of local maximum of f. Assume that $u_0 \in \mathring{U}$. Then, there exists $\varepsilon > 0$ such that $u_0 + \varepsilon q \in U$. But

$$f(u_0 + \varepsilon q) = f(u_0) + \varepsilon \|q\|^2 > f(u_0),$$

where the inequality is strict because $q \neq 0$. This is a contradiction with the local maximality of u_0. □

Lemma 1.5.15 (Number of switches)**.** *Assume that* (1.1) *is controllable. Then, for every* $z^1 \in \mathbb{R}^n$ *with* $z^1 \neq 0$ *and* $T > 0$*, the set*

$$Z = \{t \in (0,T),\, B^* z(t) = 0\}$$

is finite.

Proof. Assume that Z is infinite. Then, by analyticity of z we obtain

$$B^* z(t) = 0, \quad \forall t \in [0,T].$$

The controllability of (1.1) then implies that $z^1 = 0$ (see Theorem 1.2.1), a contradiction. □

Proof of Theorem 1.5.13. By Theorem 1.5.12, there exists $z^1 \in \mathbb{R}^n$ with $z^1 \neq 0$ such that $z(t) = e^{(T-t)A^*} z^1$ satisfies

$$B^* z(t) \cdot u(t) = \max_{u \in U} B^* z(t) \cdot u \quad \text{a.e. } t \in (0,T).$$

For every $t \in (0,T)$ and $u \in U$ we define $f_t(u) = B^* z(t) \cdot u$. Observe that $B^* z = 0$ only on a set of zero measure by Lemma 1.5.15. Therefore, the conclusion follows from Lemma 1.5.14. □

Remark 1.5.16. In the case $m = 1$ and $U = [a,b]$ $(a < b)$, we see that the function f of Lemma 1.5.14 only has one maximum, which is attained at $u = b$ if $q > 0$ and at $u = a$ if $q < 0$. Therefore, in this case, if $u \in L^2(0,T)$ is an extremal control, then, for a.e. $t \in (0,T)$,

$$u(t) = \begin{cases} b & \text{if } B^* z(t) > 0, \\ a & \text{if } B^* z(t) < 0, \end{cases}$$

for some $z^1 \neq 0$. This explains the terminology "bang-bang".

1.6 Bibliographical notes

The proof of Proposition 1.3.9 is taken from [Boy17, Chapter II, Section 2]. For additional material on constrained controllability and time-optimal problems we refer to [LM67, Chapter 2], [Son98, Section 3.6 and Chapter 4] and the references therein. Let us also mention that other type of constraints such as positivity of the controls are also discussed in [Zab08, Part I, Chapter 4]. For controllability conditions for time-dependent linear O.D.E.s we refer to [Cor07, Chapter 1]. Finally, good material for an introduction to the controllability of nonlinear systems can be found in [Cor07, Chapter 3], [Son98, Chapter 4] and [Zab08, Part II, Chapter 1].

Chapter 2

Controllability of the heat equation

2.1 Background on the heat equation

In this chapter we consider the heat equation on a nonempty bounded open subset $\Omega \subset \mathbb{R}^N$ of class C^2:

$$\begin{cases} \partial_t y - \Delta y = \mathbb{1}_\omega u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where:

- $T > 0$ is the time of control,
- y is the state y^0 is the initial data,
- u is the control,
- $\omega \subset \Omega$ localizes in space the control (we assume that ω is a nonempty open subset),
- $\mathbb{1}_\omega$ is the characteristic function of the set ω, that is the function defined by

$$\mathbb{1}_\omega(x) = \begin{cases} 1 & \text{if } x \in \omega, \\ 0 & \text{if } x \notin \omega. \end{cases}$$

We will also use the same notation to denote the operator $\mathbb{1}_\omega : L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$\mathbb{1}_\omega u(x) = \begin{cases} u(x) & \text{if } x \in \omega, \\ 0 & \text{if } x \notin \omega. \end{cases}$$

Note that $\mathbb{1}_\omega$ is a bounded linear operator with $\|\mathbb{1}_\omega\|_{\mathcal{L}(L^2(\Omega))} \leq 1$.

For a function y of (t, x) we will use the notation $y(t)$ to denote the function $y(t) : x \longmapsto y(t, x)$.

Our presentation will be based on the following fundamental result:

Theorem 2.1.1 (Spectral decomposition)**.** *There exists an orthonormal basis of $L^2(\Omega)$ formed of eigenfunctions of the Dirichlet Laplacian Δ. More precisely, there exist $\{\phi_k\}_{k\in\mathbb{N}^*} \subset H^2(\Omega) \cap H_0^1(\Omega)$ and $\{-\lambda_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}$ such that, for every $k \in \mathbb{N}^*$,*

$$\begin{cases} \Delta\phi_k &= -\lambda_k\phi_k \quad \text{in } \Omega, \\ \phi_k &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with

$$\langle \phi_k, \phi_\ell \rangle_{L^2(\Omega)} = \delta_{k\ell}, \quad \forall k, \ell \in \mathbb{N}^*, \qquad w = \sum_{k=1}^{+\infty} \langle w, \phi_k \rangle_{L^2} \phi_k, \quad \forall w \in L^2(\Omega),$$

and

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots, \qquad \lambda_k \xrightarrow[k\to+\infty]{} +\infty.$$

Let us do a little digression to motivate the notion of solution we will introduce for (2.1). Assume that $y \in C^2([0, T] \times \overline{\Omega})$ is a classical solution to (2.1) (that is, the equation, the boundary condition and the initial condition hold pointwisely). Then, we multiplying (2.1) by ϕ_k and integrating over Ω we obtain the O.D.E.

$$\begin{cases} y_k'(t) + \lambda_k y_k(t) = f_k(t), \quad t \in (0, T), \\ y_k(0) = y_k^0, \end{cases}$$

where

$$y_k(t) = \langle y(t), \phi_k \rangle_{L^2}, \quad y_k^0 = \left\langle y^0, \phi_k \right\rangle_{L^2}, \quad f_k(t) = \langle \mathbb{1}_\omega u(t), \phi_k \rangle_{L^2}.$$

Thus,

$$y_k(t) = e^{-\lambda_k t} y_k^0 + \int_0^t e^{-\lambda_k(t-s)} f_k(s)\, ds, \quad \forall t \in [0, T].$$

Since $y(t) \in L^2(\Omega)$, we have

$$y(t) = \sum_{k=1}^{+\infty} \langle y(t), \phi_k \rangle_{L^2} \phi_k.$$

Therefore,

$$y(t) = \sum_{k=1}^{+\infty} e^{-\lambda_k t} y_k^0 \phi_k + \sum_{k=1}^{+\infty} \left(\int_0^t e^{-\lambda_k(t-s)} f_k(s)\, ds \right) \phi_k.$$

Definition 2.1.2 (Mild solution)**.** Let $T>0$, $y^0 \in L^2(\Omega)$ and $u \in L^2(0,T;L^2(\Omega))$. The function $y:(0,T)\times\Omega \longrightarrow \mathbb{R}$ defined for every $t\in[0,T]$ by

$$y(t) = S(t)y^0 + \int_0^t S(t-s)\mathbb{1}_\omega u(s)\,ds, \tag{2.2}$$

where

$$S(t)y^0 = \sum_{k=1}^{+\infty} e^{-\lambda_k t}\langle y^0,\phi_k\rangle_{L^2}\phi_k,$$

is called the (mild) solution to (2.1).

Note that we have the estimate

$$\|y(t)\|_{L^2(\Omega)} \leq \left\|y^0\right\|_{L^2(\Omega)} + \sqrt{T}\,\|u\|_{L^2(0,T;L^2(\Omega))}, \quad \forall t\in[0,T]. \tag{2.3}$$

Remark 2.1.3. Formally, (2.1) can be recast as an infinite dimensional O.D.E.in the Hilbert space $H=L^2(\Omega)$:

$$\begin{cases} \dfrac{d}{dt}y &=& Ay+Bu, \quad t\in(0,T), \\ y(0) &=& y^0, \end{cases}$$

where $A:D(A)\subset H \longrightarrow H$ is the Dirichlet Laplacian operator:

$$A=\Delta, \quad D(A)=H^2(\Omega)\cap H_0^1(\Omega),$$

and $B\in\mathcal{L}(H,U)$ with control space $U=L^2(\Omega)$ is simply $B=\mathbb{1}_\omega$. Then, (2.1) is nothing but the Duhamel's formula where $S(t)$ is the generalization of the exponential of a matrix (it is the so-called semigroup generated by Δ).

All along this chapter we choose a presentation that is based on the explicit Fourier representation for the solution to the heat equation (2.2) (and of its adjoint system (**??**) below) since it does not require any particular knowledge in PDEs, semigroup nor spectral theory.

2.2 Controllability and duality

Definition 2.2.1 (Controllability)**.** We say that (2.1) is:

(i) exactly controllable in time T if, for every $y^0, y^1 \in L^2(\Omega)$, there exists $u\in L^2(0,T;L^2(\Omega))$ such that the corresponding solution y to (2.1) satisfies

$$y(T)=y^1.$$

(ii) null-controllable in time T if the above property holds for $y^1=0$.

(iii) approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in L^2(\Omega)$, there exists $u \in L^2(0,T;L^2(\Omega))$ such that the corresponding solution y to (2.1) satisfies

$$\left\| y(T) - y^1 \right\|_{L^2(\Omega)} \leq \varepsilon.$$

Remark 2.2.2. The notion of exact controllability is not relevant for the heat equation because of its regularizing effect. Indeed, in the case $\omega = \Omega$ the solution y to (2.1) with $y^0 = 0$ satisfies $y(\varepsilon) \in H_0^1(\Omega)$ as soon as $\varepsilon > 0$. It follows that a target $y^1 \in L^2(\Omega) \backslash H_0^1(\Omega)$ will never be reached.

We now proceed as in the finite dimensional case. We introduce the linear operators

$$\begin{array}{rccl} F_T & : & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & & y^0 & \longmapsto & \overline{y}(T), \end{array}$$

where $\overline{y}$ is the solution to the equation (2.1) with $u = 0$, and

$$\begin{array}{rccl} G_T & : & L^2(0,T;L^2(\Omega)) & \longrightarrow & L^2(\Omega) \\ & & u & \longmapsto & \widehat{y}(T), \end{array}$$

where $\hat{y}$ is the solution to the equation (2.1) with $y^0 = 0$. With these notations, we can restate the different notions of controllability as follows:

(i) (2.1) is null-controllable in time T if, and only if,

$$\operatorname{Im} F_T \subset \operatorname{Im} G_T.$$

(ii) (2.1) is approximately controllable in time T if, and only if,

$$\overline{\operatorname{Im} G_T} = L^2(\Omega).$$

Since F_T and G_T are bounded linear operators thanks to (2.3), by duality we obtain:

(i) (2.1) is null-controllable in time T if, and only if, (see e.g. [TW09, Proposition 12.1.2])

$$\exists C > 0, \quad \left\| F_T^* z^1 \right\|_{L^2(\Omega)}^2 \leq C^2 \left\| G_T^* z^1 \right\|_{L^2(\Omega)}^2, \quad \forall z^1 \in L^2(\Omega).$$

(ii) (2.1) is approximately controllable in time T if, and only if,

$$\ker G_T^* = \{0\}.$$

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